HUNGARY–ISRAELI MATHEMATICS COMPETITION: THE FIRST TWELVE YEARS

S GUERON

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S Gueron

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DEDICATION

I dedicate this book to the loving memory of Professor Joe Gillis (1911-1993), who was the founder of the Israeli Mathematical Olympiads, a great mathematician, and a dear friend.

Shay Gueron

FOREWORD

In my role of Executive Director of the Australian Mathematics Trust, and currently as President of the World Federation of National Mathematics Competitions I very much welcome the publication of this book.

The book meets an important need in making high quality problem material available to students who are aspiring to compete at the national and international level at Mathematics Olympiads. The material in this book is very special. There are some beautiful problems here and they should also serve to motivate and challenge the reader.

The book also makes a very strong statement about how mathematics can cross barriers. When this event started the two countries involved did not even have diplomatic relations. The founders of the event from both Hungary and Israel should be applauded for their remarkable initiative.

I have personally worked on this book as a typesetter and proof-reader. Shay Gueron supplied all of the material completely in TEX source code and I reformatted it in the AMT Publishing style, proof-reading and drawing the diagrams as I went.

I would like to thank my colleague Dr Andrei Storozhev for proof-reading the book on behalf of the publisher and making many helpful suggestions which improved the finished product.

Finally I would like to thank Shay Gueron, who has been very helpful to work with on this project.

Peter Taylor
Canberra
01 June 2003

PREFACE

The History of the Competition

The Hungary-Israel Binational Mathematical Competition commenced in 1990. In those days, official diplomatic relations between the two countries were only their infancy. It was in the summer of 1988, during the International Mathematical Olympiad (IMO) in Australia, where we, the late Professor Joe Gillis and myself, met János Pataki and József Pelikán, the Hungarian IMO team leaders. Together, we amused ourselves with the idea of initiating a binational mathematical competition between the two countries. At that time it sounded like an imaginary plan, but eventually, we made it come true.

The Hungary-Israel Binational Mathematical Competition is a competition between teams of (four to six) students from each country. It is held in the month of April each year, taking place in Israel in the even years, and in Hungary in the odd years.

The competition is supported by the Israeli Ministry of Education, the Israeli Ministry of Foreign Affairs, the Hungarian Mathematical Society (Bolyai János Matematikai Társulat) the Hungarian Ministry of Education, and the Hungarian Ministry of Foreign Affairs.

Aims and Format of the Competition

The aims of the Hungary-Israel Binational Mathematical Competition include: discovering, encouraging and challenging mathematically gifted students from the two countries, establishing friendly relations and cooperation between students and teachers, creating an opportunity for the exchange of information on school syllabi and practice, encouraging mathematical involvement with problem solving and Olympiad type activities, and serving as a good international experience for the students that prepare for the coming IMO.

The Hungary-Israel Binational Mathematical Competition is a one week event held in each year. The actual competition takes place during two or three days. In the years 1990-1996, two one-day competitions were held: an individual competition consisting of four problems, given in the first day, and a team competition given in the second day. Starting in 1997, the format was changed to an IMO style exam: two days of individual competition with 3 problems and 4.5 hours working time in each. In the year 2000, a team competition was added, as a third day, to the two individual competition days.

The team competitions concentrate on one advanced topic that is announced ahead of time, and some reference materials are decided by the

team leaders. The students learn the new topic and train for the competition.

Apart from the actual days of the competition, the rest of the week is devoted to marking the students' exams, mathematical activities, site seeing and entertainment.

The Organization of the Book

This book summarizes the first twelve years, 1990-2001, of the Hungary-Israel Binational Mathematical Competition. It includes the problems and the complete solutions to the individual and team competition problems, as well as final answers and hints to the individual competition problems. Altogether, the book includes the 58 problems that appeared in the twelve individual competitions, and the 44 problems that appeared in the eight team competitions.

A preliminary Hebrew version of this book, written by me, was published in 1996 in Israel.

Chapter 1 introduces the statements of the individual competition problems, Chapter 2 provides the final answers (where such answers exist), Chapter 3 offers hints for the solutions, and Chapter 4 provides the complete solutions. Chapter 5 presents the problem statements of the team competition, and the complete solutions appear in Chapter 6.

Some of the problems in the book are easy, and some are rather difficult. However, no special or advanced knowledge, beyond that of the typical IMO contestant's curriculum, is required in order to tackle the individual competition problems and/or to read and understand the solutions. For the convenience of the reader, we have added a Glossary section explaining the terms and theorems which are not standard, and have been used in the book. The team competition problems are typically more advanced and require some reading on the related topic.

The book is directed to the wide audience of mathematics lovers, problem solving enthusiasts, and students who wish to improve their mathematical competitions skills. I sincerely hope that the book would be helpful to all readers.

Shay Gueron
Haifa
01 June 2003

ACKNOWLEDGEMENTS

I am pleased to acknowledge the help of my friends and students who went carefully over the text and made many helpful remarks, corrections, and proposals.

Dr. Seannie Dar from the Academic College of Tel-Aviv Yafo.

Twice an IMO silver medalist, and a gold medalist in IMO 1988. Seannie has been the Israeli Deputy Team Leader for the IMO since 1993, and helped me with the organisation of the Hungary-Israel Binational Mathematical Competition in the past few years.

Eran Assaf (bronze medalist in IMO 2000, 2002, silver medalist in IMO 2001), *Mark Braverman* (bronze medalist in IMO 1998, 1999, and gold medalist in IMO 2000), *Maxim Iorsh* (bronze medalist in IMO 1993, 1994, silver medalist in IMO 1995), *Oran Lang* (bronze medalist in IMO 1998, 1999, gold medalist in IMO 2000, 2001), *Ran Tessler* (bronze medalist in IMO 1999, silver medalist in IMO 2000, 2001, 2002).

Without their help - the number of mistakes in this book (which, I am afraid, is still a positive integer) would have been larger.

In particular, it was a pleasure to work together in a team for preparing the solutions of the team competitions.

I thank Prof. Noga Alon of Tel-Aviv University, who proposed the solution to problem 5 of the 1994 team competition.

I would also like to acknowledge the help and comments of my dear Hungarian friends János Pátaki (the Hungarian team leader for this competition in 1990, 1991, and from 2000 on) and Dr. József Pelikán (the Hungarian IMO team leader and the team leader for the Binational Competition in 1992-2000).

Shay Gueron
Haifa
01 June 2003

hat there exists a pair of distinct p
at the inequality

$$x_k)^{1997} = 1996 \prod_{k=1}^{1997} x_k$$

$\Delta ABC, \Delta AB_2C_2$

$+$ [redacted] $\geq \frac{1}{R}(4\sqrt{3}+6).$

$$\sum_{k=1}^{1997} 2^{k-1}(x_k)^{1997} =$$

1. PROBLEMS

1990

1. Prove that there exist no positive integers x and y such that both $x^2 + y + 2$ and $y^2 + 4x$ are perfect squares.

2. Let ABC be a triangle where $\angle ACB = 90°$. Let D be the midpoint of BC and let E, and F be points on AC such that $CF = FE = EA$. The altitude from C to the hypotenuse AB is CG, and the circumcentre of triangle AEG is H. Prove that the triangles ABC and HDF are similar.

3. Prove that

$$\frac{1989}{2} - \frac{1988}{3} + \frac{1987}{4} - \cdots - \frac{2}{1989} + \frac{1}{1990}$$

$$= \frac{1}{996} + \frac{3}{997} + \frac{5}{998} + \cdots + \frac{1989}{1990}.$$

4. A rectangular sheet of paper with integer length sides is given. The sheet is marked with unit squares. Arrows are drawn at each lattice point on the sheet in a way that each arrow is parallel to one of its sides, and the arrows at the boundary of the paper do not point outwards.

Prove that there exists at least one pair of neighboring lattice points (horizontally, vertically or diagonally) such that the arrows drawn at these points are in opposite directions.

1991

1. Let $f(x)$ be a polynomial with integer coefficients, satisfying

$$f(0) = 11, \quad f(x_1) = f(x_2) = f(x_3) = \ldots = f(x_n) = 2002,$$

for some distinct integers $x_1, x_2, \ldots, x_n$. Find the maximal value of n.

2. The vertices of a square sheet of paper are A, B, C, D. The sheet is folded in a way that the point D is mapped to the point D' on the side BC. Let A' be the image of A after the folding, and let E be the intersection point of AB and $A'D'$. Let r be the inradius of the triangle EBD'. Prove that $r = A'E$.

3. Let H_n be the set of all numbers of the form

$$2 \pm \sqrt{2 \pm \sqrt{2 \pm \ldots \pm \sqrt{2}}}$$

where "root signs" appear n times.

   (a) Prove that all the elements of H_n are real.

   (b) Compute the product of the elements of H_n.

   (c) The elements of H_{11} are arranged in a row, and are sorted by size in an ascending order. Find the position in that row, of the element of H_{11} that corresponds to the following combination of $\pm$ signs:

$$+ + + + + - + + - + -$$

4. Find all the real values of λ for which the system of equations

$$\begin{aligned} x + y + z + v &= 0, & (1)\\ (xy + yz + zv) + \lambda(xz + xv + yv) &= 0, & (2) \end{aligned}$$

has a unique real solution.

1992

1. Let $c \neq 1$ be a real positive number, and let n be a positive integer. Prove that

$$n^2 \leq \frac{c^n + c^{-n} - 2}{c + c^{-1} - 2}.$$

2. Let S be a set of 1992 positive integers having the following property: The list of the last digits of the elements of S includes all of the digits from 0 to 9.

   Prove that there exists a nonempty subset of S such that the sum of its elements is divisible by 2000.

3. One hundred strictly increasing sequences of positive integers are given:

$$\begin{aligned} A_1 &= \{a_1^{(1)}, a_2^{(1)}, \ldots, a_k^{(1)}, \ldots\}, \\ A_2 &= \{a_1^{(2)}, a_2^{(2)}, \ldots, a_k^{(2)} \ldots\}, \\ &\vdots \\ A_{100} &= \{a_1^{(100)}, a_2^{(100)}, \ldots, a_k^{(100)} \ldots\}. \end{aligned}$$

   For any positive integer n and integers r, s such that $1 \leq r, s \leq 100$, we define the following functions:

   $f_r(n)$ - the number of elements of A_r that do not exceed n.

   $f_{r,s}(n)$ - the number of elements of $A_r \cap A_s$ that do not exceed n.

   For every n and r it is given that

$$f_r(n) \geq \frac{1}{2}n$$

   Prove that there exists a pair of distinct positive integers (r, s), such that the inequality

$$f_{r,s}(n) \geq \frac{8n}{33}$$

   holds for at least five distinct integers n between 1 and 19920.

4. The five vertices of a given convex pentagon P are lattice points. Let Q denote the convex pentagon defined by the intersection points of the five diagonals of P. Prove that there exists at least one lattice point inside or on the boundary of Q.

1993

1. For any two integers x and y the notation $x.y$ represents a number whose integral part is composed of the digits of x and its fractional part is compose of the digits of y. For example, if $x = 123$ and $y = 456$ then $x.y = 123.456$

   Find all possible solutions of the equation

$$\frac{a}{b} = b.a$$

   where a and b are relatively prime positive integers.

2. Find all the polynomials $f(x)$ with real coefficients, satisfying

$$f(x^2 - 2x) = (f(x-2))^2$$

   for every real argument x.

3. Let H be a semicircle with radius 1, and let A, B, C, D, E be five points on its boundary, in this cyclic order. Prove that

$$AB^2 + BC^2 + CD^2 + DE^2 + AB \cdot BC \cdot CD + BC \cdot CD \cdot DE < 4$$

4. A $3n \times 3n$ chessboard is given. Find the largest number of rooks which can be placed on the chessboard in such a way that each rook is taken (threatened) by no more than one rook.

1994

1. Let m and n be two distinct positive integers. Prove that there exists a real number x such that $1/3 \leq \{xn\} \leq 2/3$ and $1/3 \leq \{xm\} \leq 2/3$.

   Here, for any real number y, $\{y\}$ denotes the fractional part of y. For example $\{3.1415\} = 0.1415$

2. Let $a_1, \ldots, a_k, a_{k+1}, \ldots, a_n$ be n positive numbers $(k < n)$. Suppose that the values of $a_{k+1}, a_{k+2}, \ldots, a_n$ are fixed. Choose the values of $a_1, a_2, \ldots, a_k$ that minimize the sum

$$\sum_{i,j,\ i \neq j} \frac{a_i}{a_j}$$

3. Three given circles have the same radius and pass through a common point P. Their other points of pairwise intersections are A, B, C. We define triangle $\Delta A'B'C'$, each of whose sides is tangent to two of the three circles. The three circles are contained in $\Delta A'B'C'$. Prove that the area of $\Delta A'B'C'$ is at least nine times the area of ΔABC.

4. An "n-m society" is a group of n girls and m boys. Prove that there exist numbers n_0 and m_0 such that every $n_0 - m_0$ society contains a subgroup of five boys and five girls with the following property: either all of the boys know all of the girls or none of the boys knows none of the girls.

1995

1. Let the sum of the first n primes be denoted by S_n. Prove that for any positive integer n, there exists a perfect square between S_n and S_{n+1}.

2. Let P, P_1, P_2, P_3, P_4 be five distinct points on a circle. The distance of P from the line P_iP_k is denoted by d_{ik}. Prove that $d_{12}d_{34} = d_{13}d_{24}$.

3. The polynomial $f(x) = ax^2+bx+c$ has real coefficients and satisfies $|f(x)| \leq 1$ for all $x \in [0,1]$. Find the maximal value of $|a|+|b|+|c|$.

4. Consider a convex polyhedron whose faces are triangles. Prove that it is possible to colour its edges by either red or blue, in a way that the following property is satisfied: one can travel from any vertex to any other vertex while passing only along red edges, and can also do this while passing only along blue edges.

1996

1. Find all series of integers $x_1, x_2, \ldots, x_{1997}$ such that

$$\sum_{k=1}^{1997} 2^{k-1}(x_k)^{1997} = 1996 \prod_{k=1}^{1997} x_k \tag{1}$$

2. Let $n > 2$ be an integer. Suppose that n^2 can be represented as the difference of the cubes of two consecutive positive integers. Prove that n is the sum of two squares. Prove also that such an n really exists.

3. A given convex polyhedron has no vertex which is incident with exactly three edges. Prove that the number of faces of the polyhedron which are triangles is at least eight.

4. Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers and let $b_1, b_2, \ldots, b_n$ be arbitrary real numbers satisfying $1 \geq b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$. Prove that there exists a positive integer $k \leq n$ for which the inequality $|a_1b_1 + a_2b_2 + \ldots + a_nb_n| \leq |a_1 + a_2 + \ldots + a_k|$ holds.

1997

1. Is there an integer N such that

$$(\sqrt{1997} - \sqrt{1996})^{1998} = \sqrt{N} - \sqrt{N-1}?$$

2. Find all the real numbers α that satisfy the following property: for any positive integer n there exists an integer m such that

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{3n}$$

3. Let ABC be an acute angled triangle whose circumcentre is O. The three diameters of the circumcircle that pass through A, B, and C, meet the opposite sides BC, AC and AB at the points A_1, B_1. C_1, respectively.

The circumradius of ABC is of length $2P$, where P is a prime number. The lengths OA_1, OB_1, OC_1 are integers. What are the lengths of the sides of the triangle?

4. What is the number of distinct sequences of length 1997 that can be formed by using the letters A, B, C, where each letter appears an odd number of times?

5. Let ABC be a given triangle. The three squares

$$ACC_1A'', \qquad ABB_1A', \qquad BCDE$$

are constructed, outwards, on the sides of ABC. The centre of the square $BCDE$ is denoted by P. Prove that the three lines $A'C$, $A''B$ and PA pass through one point.

6. Can a closed disk be decomposed into a union of two congruent parts having no common points?

1998

1. A player plays the following game: in each turn he flips a fair coin and guesses the side on which it would fall. A successful guess credits the player with one point. In any unsuccessful guess the player loses all the points he has accumulated. The player starts the game with 0 points. The game terminates when the player has accumulated 2 points.

   I. What is the probability p_n that the game terminates after exactly n turns?

   II. What is the expected (average) number of turns until the game terminates?

2. The circumcentre of an acute angled triangle ABC is O, and its circumradius is R. The radii of the incircles of the triangles OBC, OCA, OAB are r_1, r_2, r_3, respectively. Prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3} \geq \frac{1}{R}(4\sqrt{3}+6).$$

3. Let a, b, c, m, n be positive integers, and $f(x) = ax^2+bx+c$. Prove that there exist n consecutive positive integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that each one of the n integers $f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n)$ has at least m different prime factors.

4. Find all the positive integers x and y such that $5^x - 3^y = 16$.

5. Definition: A hexagon is called *affinely regular* if it is centrally symmetric and the three pairs of its opposite sides are parallel to the diagonal joining the two remaining vertices.

   Let $ABCDEF$ be a convex hexagon. Six equilateral triangles are constructed, outwards, on its sides. Prove that the third vertices of these triangles are vertices of a regular hexagon if and only if the original hexagon $ABCDEF$ is *affinely regular.*

6. Let n be a positive integer. A partition of n is a decomposition of n into a sum of positive integers. Two partitions are not considered to be different if they differ only in the order of the summands. The set of all the partitions of n is denoted by Π. If α is a partition of n, we denote the numbers of terms $1, 2, \ldots, n$ in it by $a_1(\alpha), a_2(\alpha), \ldots, a_n(\alpha)$, respectively. Prove that

$$\sum_{\alpha\in\Pi} \frac{1}{1^{a_1(\alpha)}a_1(\alpha)! \cdot 2^{a_2(\alpha)}a_2(\alpha)! \cdots n^{a_n(\alpha)}a_n(\alpha)!} = 1.$$

1999

1. Let $f(x)$ be a polynomial whose degree is at least 2. Define the sequence $g_i(x)$ by: $g_1(x) = f(x)$ and $g_{n+1}(x) = f(g_n(x))$ for $n = 1, 2, \ldots$. Let r_n be the average of the roots of $g_n(x)$. It is given that $r_{19} = 99$. Find r_{99}.

2. A set of $2n + 1$ lines in a plane is drawn. No two of them are parallel, and no three pass through one point. Every three of these lines form a non-right triangle. Determine the maximal number of acute angled triangles, that can be formed.

3. Find all the functions f from the set of rational numbers to the set of real numbers, such that for every rational x, y
$$f(x+y) = f(x)f(y) - f(xy) + 1.$$

4. Let c be a positive integer. Define the following sequence:
$$a_1 = c, \qquad a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-1)}, \quad n = 1, 2, \ldots.$$
Prove that all the terms a_n are positive integers.

5. The function
$$f(x, y, z) = \frac{x^2+y^2+z^2}{x+y+z}$$
is defined for every x, y, z such that $x + y + z \neq 0$. Find a point (x_0, y_0, z_0) such that
$$0 < x_0^2 + y_0^2 + z_0^2 < 1/1999$$
and
$$1.999 < f(x_0, y_0, z_0) < 2.$$

6. An exam consists of four multiple choice questions, where each question has three choices. A certain group of examinees took the exam. It turned out that for any subset of three of examinees, there was at least one question to which their choices covered all three possibilities. What is the maximal number of examinees in this group?

2000

1. Let S be the set of all partitions of 2000. For every partition p in S we define the function

$$f(p) = (\text{number of summands in } p) + (\text{maximal summand in } p)$$

Compute the minimal value that $f(p)$ attains over S.

Remark: A partition of a positive integer n is its representation as a sum of positive integers. Two partitions differing from each other only by order of the summands are not considered to be different.

2. Prove or disprove the following claim: For any positive integer k, there exists a positive integer $n > 1$ such that the binomial coefficient $\binom{n}{i}$ is divisible by k for any $1 \leq i \leq n-1$.

3. Let ABC be a triangle which is not equilateral. Let A_1, B_1 and C_1 be the touching points of the incircle of $\triangle ABC$ on the corresponding sides, and M the orthocenter of triangle $A_1B_1C_1$. Prove that M is on the straight line through the incentre and circumcentre of triangle ABC.

4. Let $S = \{1, 2, \ldots, 2000\}$. Consider two sets $A, B \subseteq S$, such that $|A| \cdot |B| \geq 3999$. Prove that $(A - A) \cap (B - B)$ is nonempty.

The notation $X - X$ stands here for: $X - X = \{s - t | s, t \in X, s \neq t\}$.

The notation $|X|$ stands for the number of elements in X.

5. d is a given integer. Let S be the set: $S = \{m^2 + dn^2 \mid m, n \in Z\}$. Suppose that p, q are two elements of S, and that p is prime. Suppose further that $r = \frac{q}{p}$ is an integer. Prove that r also belongs to S.

6. k and l are two given positive integers and a_{ij}, $1 \leq i \leq k$ and $1 \leq j \leq l$, are kl given positive numbers.

Prove that if $q \geq p > 0$ then

$$\Big(\sum_{j=1}^{l}\Big(\sum_{i=1}^{k} a_{ij}^{p}\Big)^{q/p}\Big)^{1/q} \leq \Big(\sum_{i=1}^{k}\Big(\sum_{j=1}^{l} a_{ij}^{q}\Big)^{p/q}\Big)^{1/p}.$$

2001

1. Find positive integers x, y, z such that

$$x > z > 1999 \cdot 2000 \cdot 2001 > y$$

satisfying

$$2000x^2 + y^2 = z^2.$$

2. A, B, C and D are points lying on the line l in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(f(x)) = f(x) + x$$

for all real x.

4. Let $P(x) = x^3 - 3x + 1$. Find the polynomial Q whose roots are the fifth power of the roots of P.

5. $\triangle ABC$ is a given triangle. B_1, C_1 are the midpoints of AC, AB respectively. I is the incentre of $\triangle ABC$. The lines B_1I and C_1I meet AB and AC in C_2 and B_2 respectively.

   Given that the areas of the triangles $\triangle ABC$ and $\triangle AB_2C_2$ are equal, what is $\angle BAC$?

6. 32 positive integers, which sum up to 120 and none of which is greater than 60 are given.

   Prove that they can be divided into two distinct subsets that have equal sum.

2. ANSWERS

1991

1. $n = 4$ 3. (b) 2 (c) 1991 4. $\frac{3-\sqrt{5}}{2} \leq \lambda \leq \frac{3+\sqrt{5}}{2}$

1993

1. $a = 5$, $b = 2$. 2. $f(x) = (x+1)^k$, for any positive integer k or $f(x) = 0$ for all x. 4. $4n$

1994

2. $a_1 = a_2 = \ldots = a_k = \sqrt{ab}$ where

$$a = a_1 + \cdots + a_n, \quad b = \frac{1}{a_1} + \cdots + \frac{1}{a_n}.$$

4. for example $n_0 = 9$ and $m_0 = 1 + 4 \times 2^9$.

1995

3. 17

1996

1. Only the trivial solution $x_1 = x_2 = \ldots = x_{1997} = 0$.

1997

1. Yes 2. The set of all integers. 3. $a = b = c = 2P\sqrt{3}$
4. $(3^{1997} - 3)/4$. 6. No.

1998

1. I. $p_n = \frac{1}{2^n} F_{n-1}$, $n = 2, 3, \ldots$, where F_n is the n-th term of the Fibonacci sequence $(1, 1, 2, 3, 5, 8, \ldots)$. II. $x = 6$ 4. $x = y = 2$.

1999

2. $\binom{2n+1}{3} + \binom{n}{2}(2n+1)$ 3. $f(x) = 1$ or $f(x) = x + 1$.
5. For example $(t, t^2 - t, 0)$ with $t = 1/10000$. 6. 9

2000

1. 90. 2. The claim is false.

2001

1. $(x, y, z) = (90000^2 - 2000 + 2 \cdot 90000, 90000^2 - 2000 - 2 \cdot 2000 \cdot 90000, 2000 + 90000^2)$.
2. The locus is the segment BC, the points in l outside AD, and the perpendicular bisector of BC if $AB = CD$ or the circle of inversion that maps A, B to D, C, if $AB \neq CD$.
3. $f(x) = \frac{1+\sqrt{5}}{2}x$ or $f(x) = \frac{1-\sqrt{5}}{2}x$. 4. $Q(x) = x^3 + 15x^2 - 198x + 1$.
5. $\angle BAC = 60°$.

3. HINTS

1990

1. Prove that $4x - 2 \leq 2y \leq 4x - 1$, which implies that $y = 2x - 1$. Since $x^2 + y + 2$ is a perfect square, you can conclude that $(2x)^2 < k^2 < (2x+1)^2$, which yields a contradiction.

2. Choose a coordinate system where C is the origin, AC is along the y-axis, BC is along the x-axis, and $AC = 1$. Express the coordinates of the vertices A, B, C, D, F, G, H and lengths of the sides in terms of $\angle CAB$.

3. Prove first the identity

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1$$

for example, by induction on n.

4. Consider closed arrow-paths, and rotate all the arrows in such path by 90°.

1991

1. Define a new polynomial $g(x) = f(x) - 2002$. Use the factorization of 1991 ($= -g(0)$) into two distinct prime factors.

2. Note that if the sides of a right angled triangle are a an b, the hypotenuse is c, and r is the inradius, then $r = \frac{1}{2}(a + b - c)$.

3. (a) Define a sequence of polynomials P_n by

$$P_0 = x - 2, \qquad P_n = P_{n-1}^2 - 2, \quad n = 1, 2, \dots \tag{1}$$

and deduce that H_n is the set of all the roots of P_n.

(b) Use (1) to show that $P_n(0) = 2$ for $n > 0$.

(c) H_{11} has $2^{11} = 2048$ elements. Suppose that $x \in H_{11}$ is the number that corresponds to the combination $+ + + + + - + + - + -$ of $\pm$ signs.

Since x starts with "$2 + \sqrt{}$", it follows that x is larger than all the elements of H_{11} that start with "$2 - \sqrt{}$". Therefore, x belongs to the list of the 1024 largest elements of H_{11}.

4. The trivial solution $(x, y, z, v) = (0, 0, 0, 0)$ always exists. For other solutions, try a solution of the form $(1, P, -P, 1)$ for some P. Using (2), derive a quadratic equation for P, and find the values of λ for which the system has real solutions of this type. This would give you the range where there is no unique solution.

1992

1. Write

$$c^n + c^{-n} - 2 = (d^n - d^{-n})^2, \qquad c + c^{-1} - 2 = (d - d^{-1})^2,$$

where $d = \sqrt{c}$, and use the Arithmetic Mean - Geometric Mean inequality.

2. Choose ten elements $a_1, a_2, \ldots, a_{10} \in S$ with different unit digits and consider the set

$$A = \{a_1, a_2, \ldots, a_{10}, T - a_1, \ldots, T - a_{10}\}$$

Where $T = a_1 + a_2 \ldots + a_{10}$.

Show that there are 19 elements in A that have different residues modulo 2000.

3. Check small values of n.

4. For a given pentagon, construct the smallest sub-pentagon which yields a counterexample. Note that in every convex pentagon whose vertices are lattice points, one of the midpoints of its sides or its diagonals is also a lattice point.

1993

1. Suppose that $10^{n-1} \leq a \leq 10^n$ for some n. You can express the decimal representation $b.a$ as $\frac{10^n b + a}{10^n}$. This number equals $\frac{a}{b}$. Conclude from this information that ab and $a - b^2$ are relatively prime.

2. Denoting $y = x - 1$, you can write $f(y^2 - 1) = [f(y-1)]^2$. Show that if $y_0 - 1$ is a root of f, then so is $y_1 - 1$, where y_1 is a square root of y_0.

3. Prove the stronger inequality

$$AB^2 + BC^2 + CD^2 + DE^2 + AB \cdot BC \cdot CD + BC \cdot CD \cdot DE < 4$$

by assuming that AE is a diameter, and by using the cosine law.

4. Divide the chessboard into n 3×3 subboards with squares a_{ij}, $1 \leq i, j \leq 3$. In each such subboard put rooks in the squares $a_{11}, a_{12}, a_{23}, a_{33}$. Count the maximal number of rooks in two ways.

1994

1. Define sequences I_i and J_i of intervals such that for any $x \in I_i$

$$\frac{1}{3} \le \{nx\} \le \frac{2}{3}$$

and similarly for J_i.

2. Apply the Cauchy-Schwartz inequality to the pairs

$$(\sqrt{x}, \sqrt{y}), \quad (\sqrt{a}, \sqrt{b}),$$

where

$$\begin{aligned} x &= a_1 + \cdots + a_k, \\ y &= \frac{1}{a_1} + \cdots + \frac{1}{a_k}, \\ a &= a_{k+1} + \cdots + a_n, \\ b &= \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}. \end{aligned}$$

3. Denote the centres of the circles by A'', B'', C'', and their common radius by R. The points A', B', C' bisect $B''C''$, $C''A''$, and $A''B''$, respectively. Use the similarity relation

$$\triangle ABC \sim \triangle A'B'C' \sim \triangle A''B''C''$$

and the inequality $R \ge 2r$ (where r is the inradius of $A'B'C'$) to obtain

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'} \ge 3.$$

4. Choose $n_0 = 9$, $m_0 > 4 \cdot 2^9$ and use the pigeonhole principle.

1995

1. Note that there is a square between any two integers a and b if $\sqrt{a} - \sqrt{b} > 1$, i.e. $a - b > 1 + 2\sqrt{b}$. If p_n denotes the n-th prime, it is sufficient to show that

$$p_{n+1} = S_{n+1} - S_n > 1 + 2\sqrt{S_n}.$$

2. Denote the radius of the circle by R. Use the formula

$$S_{\triangle ABC} = \frac{abc}{4R}$$

for the area of a triangle $\triangle ABC$ whose sides are a, b, c and its circumradius is R. Apply it to $\triangle PP_1P_2$, $\triangle PP_3P_4$, $\triangle PP_1P_3$ and $\triangle PP_2P_4$.

3. Substitute $x = 0, \frac{1}{2}, 1$ in the given inequality $|f(x)| \leq 1$, and obtain three inequalities involving a, b, c. Express a and b in terms of $A = a+2b+4c$ and $B = a+b+c$. Use triangle inequality to obtain the upper limit of 17. To show that this maximal value is attained, consider the polynomial $f(x) = 8x^2 - 8x + 1$.

4. Choose any vertex C of the polyhedron. Connect it with its neighboring vertices $C_1, \ldots, C_n$. colour the sides as follows: CC_1 blue, $CC_2, CC_3, \ldots, CC_n$ red, C_1C_2 red, $C_2C_3, C_3C_4, \ldots, C_{n-1}C_n, C_nC_1$ blue.

1996

1. Note that if

$$(x_1, x_2, \ldots, x_{1997})$$

is a solution, then so is

$$\left(\frac{x_1}{2}, \frac{x_2}{2}, \ldots, \frac{x_{1997}}{2}\right).$$

So, you can divide any solution by 2 until at least one of the x_is is odd.

2. Denote the consecutive integers by b and $b+1$, and obtain

$$3(2b+1)^2 = (2n+1)(2n-1)$$

Now, consider all possibilities for $2n+1$ and $2n-1$.

3. Denote the number of vertices, edges and faces by v, e, f, respectively, the number of vertices of degree i by v_i, the number of faces with i edges by f_i. Apply Euler's formula, namely

$$f + v - e = 2.$$

4. Use Abel's summation formula (summation by parts),

$$\sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{n}\left(\sum_{i=1}^{j} a_i\right)(b_j - b_{j+1}).$$

Apply the triangle inequality.

1997

1. Use the binomial expansion of

$$\left(\sqrt{x+1}-\sqrt{x}\right)^n$$

and of

$$\left(\sqrt{x+1}+\sqrt{x}\right)^n.$$

Note that

$$\left(\sqrt{x+1}-\sqrt{x}\right)\times\left(\sqrt{x+1}+\sqrt{x}\right)=1.$$

2. First show that if α is an integer the condition can be satisfied. Then assume that α is *not* an integer and that the condition is satisfied. Consider the two approximations $m_k/2^k$ and $m_{k+1}/2^{k+1}$ for some positive integer k.

3. Denote $AB=c$, $AC=b$, $BC=a$ and $OA_1=a_1$, $OB_1=b_1$, $OC_1=c_1$. Prove the identity

$$\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma=2\sin\alpha\sin\beta\sin\gamma$$

and deduce that

$$\frac{OA_1}{AA_1}+\frac{OB_1}{BB_1}+\frac{OC_1}{CC_1}=1.$$

Then, translate this equality into a quadratic equation in p.

4. Suppose that the (odd) number of As in the sequence is a. The number a ranges between 1 and 1995. Once a is chosen, there are $1996-a$ possibilities to choose the (odd) number b of Bs in the sequence. The numbers a and b determine the number of Cs. Sum up the appropriate possibilities and use the relations between the binomial coefficients.

5. Rotate ABB_1A' by 45° around B, and dilate by the factor of $\sqrt{2}$.

6. Suppose that the disk C can be decomposed into the two congruent parts $P1$ and $P2$. The centre can be in only one of these parts. Consider the congruence mapping that maps $P1$ to $P2$ and consider its inverse.

1998

1. I. Write p_n as

$$p_n = \frac{1}{2^n} M_n$$

and prove that for $n \geq 2$ $M_n = F_{n-1}$, where F_n is the n-th term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$.

II. Use Binet's formula

$$F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

to compute the average length of the game, which is defined by

$$x = \sum_{n=1}^{\infty} n p_n$$

2. Prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$$

$$= \frac{1}{R}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} + \frac{1}{\cos\gamma} + \frac{2}{\sin 2\alpha} + \frac{2}{\sin 2\beta} + \frac{2}{\sin 2\gamma}\right)$$

and

$$\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}, \quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \frac{3\sqrt{3}}{2}.$$

Then apply the Cauchy-Schwartz inequality.

3. Prove by induction on m that if one of the numbers $f(\alpha_i)$ has at least m distinct prime factors, then the number $f(\beta_j)$, where

$$\beta_j = \alpha_j + [f(\alpha_1)f(\alpha_2)\ldots f(\alpha_n)]^2,$$

has at least $m+1$ prime factors.

4. Prove that x and y must be even, i.e. $x = 2a$ and $y = 2b$ where a, b, are natural numbers. Write $5^x - 3^y$ in the form

$$5^x - 3^y = (5^a)^2 - (3^b)^2 = (5^a + 3^b)(5^a - 3^b) = 16$$

and check all the possibilities for a and b.

5. Denote the centre of symmetry of the hexagon $ACBDEF$ by O. Use the fact that $BC \parallel AO \parallel OD$ and show that the quadrilateral $AOBC$ is a parallelogram. Show that the triangles OAB and OBC are congruent.

6. Consider the generating function $\prod_{n=1}^{\infty} e^{\frac{x^n}{n}}$.

1999

1. Use Viète formulas.

2. Assume that all the lines pass through the origin (with no loss of generality). Consider a pair of lines l_1, l_2 forming the angles α and $180° - \alpha$ between them. Assume that $r \leq n-1$ of the given $2n-1$ lines are in the domain of the angle α and $2n-r-1$ of them are in the domain of the angle $180° - \alpha$. One of the angles α and $180° - \alpha$ is obtuse.

3. Find $f(0)$ and $f(1)$ and express $f(\frac{p}{q})$ in terms of $f(1)$.

4. Prove that $a_{n+1} = 2ca_n - a_{n-1}$ for every $n \geq 2$.

5. Consider a point P of the form $P = (t, t^2 - t, 0)$ for some t.

6. Use the pigeonhole principle to show that the number of students cannot exceed 9. To show the feasibility for 9 students, construct an explicit example.

2000

1. Denote the number of summands in s by $N(s)$, and the maximal summand in s by $M(s)$. Prove that $M(s) \times N(s) \geq 2000$, and conclude that $f(s) > 89$.

2. Take the case $k = 4$. Assume by contradiction that there is an integer n such that $\binom{n}{i}$ is divisible by 4 for $1 \leq i \leq n-1$, and consider the sum of those binominal coefficients.

3. Let A_2, B_2, C_2 be the midpoints of the arcs B_1C_1, C_1A_1, A_1B_1 of the inscribed circle. Prove that $A_2B_2 \| AB$, $B_2C_2 \| BC$, $C_2A_2 \| CA$. Consider the homothety taking the inscribed circle into the circumcircle.

4. Look at the function $F: A \times B \to \{2, 3, \ldots, 4000\}$ defined by

$$F(a, b) = a + b.$$

5. Let $p = x^2 + dy^2$, $q = a^2 + db^2$. Prove that $p|(ax + by)(ax - by)$, and deduce that p divides one of $ax + by, ax - by$.

   Take
$$v = \frac{ax \pm by}{p}$$
   integer, and find an integer u such that
$$\frac{q}{p} = u^2 + dv^2.$$

6. Prove the inequality by induction on k. Note that the case $k = 2$ is simply Minkowsky's inequality.

2001

1. Use the identity $n(m^2-n+2m)^2+(m^2-n-2mn)^2=(n+1)(m^2+n)^2$.

2. Let P be a point on the locus, and consider the circles PAB, PCD. Prove that they are invariant under the reflection with respect of the perpendicular bisector (if $AB=CD$) or with the inversion which maps A, D to D, C.

3. Prove that f is $1-1$ and monotonic. Consider two cases: if f is decreasing, prove that $|f(x)|<|x|$. If f is increasing, prove that $|f^{-1}(x)|<|x|$.

4. Let a be a root of P. Then $-1=a^3-3a$. Raise to the fifth power.

5. Use the theorem of Menelaus several times to find AB_2, AC_2.

6. Prove that for $3\le i\le 30$, $a_i\le i$ when a_i is the i-th term in the sequence (by order of magnitude).

4. SOLUTIONS

1990

1. Note that $x^2 + y + 2 > x^2$. Therefore, if $x^2 + y + 2$ is a perfect square, then

$$x^2 + y + 2 \geq (x+1)^2 = x^2 + 2x + 1 \Rightarrow y \geq 2x - 1.$$

Similarly,

$$y^2 + 4x \geq (y+1)^2 = y^2 + 2y + 1 \Rightarrow 2y \leq 4x - 1.$$

In particular, $4x - 2 \leq 2y \leq 4x - 1$ and therefore $y = 2x - 1$. Since $y^2 + 4x$ is a perfect square, there exists an integer k such that

$$k^2 = (2x-1)^2 + 4x = 4x^2 + 1.$$

The last equality implies that $k^2 < (2x+1)^2$ and $k^2 > (2x)^2$, which is a contradiction.

2. Consider a coordinate system where C is the origin, AC lies on the y-axis, BC lies on the x-axis and $AC = 1$. Denote $\angle CAB = \alpha$.

The equation of the line AB is $y = -(\cot\alpha)x + 1$. The equation of the line CG is $y = (\tan\alpha)x$. Therefore, the coordinates of G are $(\sin\alpha\cos\alpha, \sin^2\alpha)$.

The point H lies on the perpendicular bisector of AE. Therefore, its y-coordinate is $\frac{5}{6}$. If x_H is the x-coordinate of H, then

$$HA^2 = x_H^2 + \left(\frac{5}{6} - 1\right)^2,$$

$$HG^2 = (x_H - \sin\alpha\cos\alpha)^2 + \left(\frac{5}{6} - \sin^2\alpha\right)^2.$$

Since $HA = HG$ we have $x_H = \frac{1}{3}\cot\alpha$.

We can now compute the sides of $\triangle HDF$. A straightforward calculation yields

$$HD^2 = \frac{4\cot^2\alpha + 9\tan^2\alpha + 13}{36},$$

$$HF^2 = \frac{4\cot^2\alpha + 9}{36},$$

and

$$FD^2 = \frac{9\tan^2\alpha + 4}{36}.$$

The sides of $\triangle ABC$ are $AB^2 = 1+\tan^2\alpha$, $BC^2 = \tan^2\alpha$, $AC^2 = 1$.

Now,

$$\left(\frac{HF}{FD}\right)^2 = \frac{4\cot^2\alpha + 9}{9\tan^2\alpha + 4} = \cot^2\alpha = \left(\frac{AC}{BC}\right)^2$$

and consequently

$$\frac{HF}{FD} = \frac{AC}{BC}.$$

Further, we also have

$$HF^2 + FD^2 = \frac{4\cot^2\alpha + 9\tan^2\alpha + 13}{36} = HD^2,$$

By the inverse Pythagorean theorem, this implies that $\triangle HDF$ is a right angled triangle. Since both $\triangle ABC$ and $\triangle HDF$ are right angled triangles and the ratios of their sides are equal, it follows that they are similar.

3. We first prove, the following general statement:

$$S_n \equiv \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{3} + \cdots + \frac{1}{2n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1. \quad (1)$$

The proof goes by induction. Verification for $n = 1$ is trivial. Now, assume that $S_k = 1$ for some k. For S_{k+1} we have:

$$\begin{aligned} S_{k+1} &= S_k - \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} + \frac{1}{2k+1} + \frac{1}{2k+2} = \\ &= S_k + \frac{1}{2k+2} + \frac{1}{2k+2} - \frac{1}{k+1} = S_k = 1. \end{aligned}$$

This completes the proof of (1).

We now return to the required equality. Its left hand side is

$$
\begin{aligned}
L &= \frac{1989}{2} - \frac{1988}{3} + \frac{1987}{4} - \cdots + \frac{1}{1990} \\
&= \left(\frac{1989}{2} + 1\right) - \left(\frac{1988}{3} + 1\right) + \cdots + \left(\frac{1}{1990} + 1\right) - 1 \\
&= 1991\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{1}{1990}\right) - 1,
\end{aligned}
$$

and its right hand side is

$$
\begin{aligned}
R &= \frac{1}{996} + \frac{3}{997} + \frac{3}{998} + \cdots + \frac{1989}{1990} = \\
&= \left(\frac{1}{996} - 2\right) + \left(\frac{3}{997} - 2\right) + \left(\frac{5}{998} - 2\right) + \cdots \\
&\quad \cdots + \left(\frac{1989}{1990} - 2\right) + 2(1990 - 996 + 1) = \\
&= 1990 - 1991\left(\frac{1}{996} + \frac{1}{997} + \frac{1}{998} + \cdots + \frac{1}{1990}\right).
\end{aligned}
$$

Therefore, using (1) we have

$$
\begin{aligned}
R - L &= 1991 - 1991\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots \right. \\
&\qquad \left. \cdots + \frac{1}{1990} + \frac{1}{996} + \frac{1}{997} + \cdots + \frac{1}{1990}\right),
\end{aligned}
$$

which completes the proof.

Alternative solution

$$
\begin{aligned}
L &= \frac{1989}{2} - \frac{1988}{3} + \ldots + \frac{1}{1990} \\
&= \left(\frac{1989}{2} + 1\right) - \left(\frac{1988}{3} + 1\right) + \ldots + \left(\frac{1}{1990} + 1\right) - 1 \\
&= 1991\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots + \frac{1}{1990}\right) - 1
\end{aligned}
$$

$$
\begin{aligned}
&= 1991\left(2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{1990}\right)\right. \\
&\quad \left.-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{1990}\right)\right)-1 \\
&= 1991\left(\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{995}\right)\right. \\
&\quad \left.-\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{1990}\right)\right)-1 \\
&= 1991\left(1-\left(\frac{1}{996}+\frac{1}{997}+\ldots+\frac{1}{1990}\right)\right)-1 \\
&= 1990-1991\left(\frac{1}{996}+\frac{1}{997}+\ldots+\frac{1}{1990}\right).
\end{aligned}
$$

Finally, L is exactly R from the first solution.

4. Let us assume that there exists no pair of neighboring points whose arrows point in opposite directions ("conflict"), and start our solution with a few definitions.

A "subdomain" as a connected region of the sheet whose boundary is composed of only line segments which are parallel to one of its sides, and connect two neighboring lattice points.

We call every subdomain whose arrows do not conflict, a "good domain". Thus, our assumption implies that every subdomain is good. Finally, we call every good domain whose boundary arrows point inwards, an "excellent domain".

We now choose any lattice point x_1 of the sheet. We construct the orbit of lattice points $x_1, x_2, x_3, \ldots$ by going from one point x_k to the next point x_{k+1} in the direction of the arrow at x_k. We continue the process until the first time that orbit crosses itself (i.e., the first time we return to a point that is already on our orbit). Since the total number of lattice points on the sheet is finite, our process ends within a finite number of steps. We end up having a closed orbit which we denote by L_1, which encloses a subdomain which we denote by A_1. Clearly, A_1 is a good domain. With no loss of generality, we may assume that L_1 goes clockwise, and the domain A_1 is found to its right.

We now rotate all the arrows on the sheet clockwise by 90°. After this rotation, all of the arrows along L_1 point inwards, and it follows that A_1 is now an excellent domain.

In the second step, we choose a lattice point x_2 that is found in the domain A_1 (if there exists such), repeat the above process and obtain a new excellent domain A_2 which is strictly included in A_1 (because the new closed orbit cannot reach L_1, the boundary of the excellent subdomain A_1).

We continue this process as long as there are still inner points in the current excellent subdomain. Finally, we end up with an excellent rectangular subdomain, say A_r, whose "width" is one square. The subdomain A_r is encircled by a closed orbit L_r of arrows that point in the directions of one of its sides. Obviously, along L_r there exist two neighboring squares with arrows at opposite directions. This contradicts our starting assumption, and the desired statement follows.

1991

1. We define $g(x) = f(x) - 2002$. If $f(x_i) = 2002$ then x_i is a root of $g(x)$. Consequently, we can write

$$g(x) = \prod_{i=1}^{n} (x - x_i) q(x) \tag{2}$$

for some polynomial $q(x)$ with integer coefficients, where n is the required maximal possible number of x_is.

Substituting $x = 0$ in (2) we obtain

$$g(0) = -1991 = -11 \times 181 = \prod_{i=1}^{n} (-x_i) q(0).$$

Since 11, and 181 are primes, it follows that -1991 can be written as the product of at most 4 different factors, namely $-1 \times 1 \times 11 \times 181$, and we therefore have $n \leq 4$.

It remains to provide an example where $n = 4$. This occurs for example with

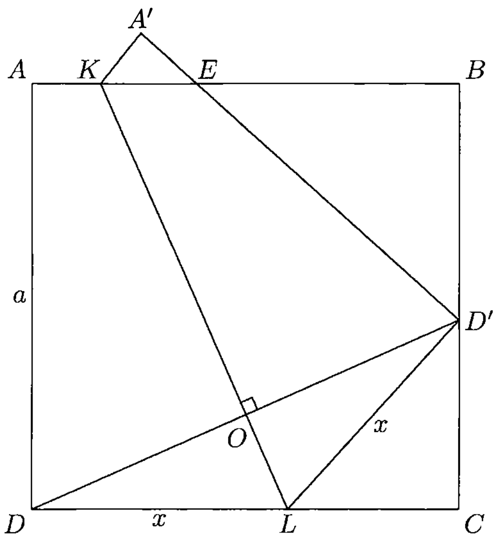
$$g(x) = (x - 1)(x + 1)(x + 11)(x + 181), \qquad q(x) = 1,$$

that is,

$$f(x) = (x - 1)(x + 1)(x + 11)(x + 181) + 2002$$

One can now easily verify that indeed $f(0) = 11$ and $f(1) = f(-1) = f(-11) = f(-181) = 2002$, as required.

2. Denote the length of the side of the square by a.



The distance of D' from AD is a. Therefore, the distance of D from ED' is also a. It follows that D is the centre of the excircle of the triangle $\triangle EBD'$. Consequently, as the lengths of the two tangents to a given circle, extended from a given point, are equal, it follows that the perimeter of $\triangle EBD'$ is $2CD = 2a$ Finally, in a right angle triangle $\triangle EBD'$ we have

$$r = \frac{1}{2}(BE + BD' - ED'),$$

that is,

$$r = \frac{1}{2}(BE+BD'+ED'-2ED') = \frac{1}{2}(2a-2ED') = a-ED' = A'E.$$

as required.

3. We prove part (a) by induction. For $n = 1$ the statement is trivial. Assume now that the elements of H_{n-1} are real and belong to the interval $(0, 4)$. Now, from the definition of H_n, it follows that if $x \in H_n$ then $(x-2)^2 \in H_{n-1}$. Using the induction hypothesis, we conclude that the elements of H_n are also real and are found in the interval $(0, 4)$.

To prove part (b) we note that the elements of H_n are the roots of the polynomial P_n, where the sequence P_n is defined by

$$P_0 = x - 2, \quad P_n = P_{n-1}^2 - 2, \quad n = 1, 2, \ldots$$

Therefore, if we assume by induction that $P_{n-1}(0) = 2$ for some $n \geq 1$ we immediately have that $P_n(0) = 2$ as well. Using Viète's formulas, it follows that the product of the roots of P_n, which is also the product of the elements of H_n, is 2.

For part (c), we note that H_{11} has $2^{11} = 2048$ elements. Let $x \in H_{11}$ denote the number that corresponds to the combination $+ + + + + - + + - + -$ of $\pm$ signs.

Since x starts with "$2+\sqrt{\quad}$", it follows that x is larger than all the elements of H_{11} that start with "$2 - \sqrt{\quad}$". Therefore, x belongs to the list of the 1024 largest elements of H_{11}. Further, the first five $+$ signs appearing in x imply that x is among the first 64 elements of H_{11}. Since these four $+$ signs are followed by a $-$ sign, it follows that x is between the 33rd and 64th largest elements of H_{11}. Finally, it is easy to conclude that x stands in the 1991^{st} position in the ascending row of the elements of H_{11}.

Alternative solution:

$$\begin{aligned}
&\left(2 + \sqrt{2 + \epsilon_n\sqrt{2 + \epsilon_{n-1}\sqrt{\ldots + \epsilon_1\sqrt{2}}}}\right) \\
&\qquad \cdot \left(2 - \sqrt{2 + \epsilon_n\sqrt{2 + \epsilon_{n-1}\sqrt{\ldots + \epsilon_1\sqrt{2}}}}\right) \\
= {}& 2^2 - \left(\sqrt{2 + \epsilon_n\sqrt{2 + \epsilon_{n-1}\sqrt{\ldots + \epsilon_1\sqrt{2}}}}\right)^2 \\
= {}& 4 - \left(2 + \epsilon_n\sqrt{2 + \epsilon_{n-1}\sqrt{\ldots + \epsilon_1\sqrt{2}}}\right) \\
= {}& 2 - \epsilon_n\sqrt{2 + \epsilon_{n-1}\sqrt{\ldots + \epsilon_1\sqrt{2}}}.
\end{aligned}$$

Hence

$$\prod_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n,\epsilon_{n+1}\in\{-1,1\}} (2+\epsilon_{n+1}\sqrt{2+\epsilon_n\sqrt{\ldots}})$$

$$= \prod_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n\in\{-1,1\}} (2-\epsilon_n\sqrt{2+\epsilon_{n-1}\sqrt{\ldots}})$$

$$= \prod_{\epsilon_1,\epsilon_2,\ldots,\bar{\epsilon}_n\in\{-1,1\}} (2+\bar{\epsilon}_n\sqrt{2+\epsilon_{n-1}\sqrt{\ldots}}).$$

Therefore,

$$\prod_{x\in H_{n+1}} x = \prod_{x\in H_n} x = \ldots = \prod_{x\in H_0} x = 2.$$

4. We first note that the trivial solution $x = y = z = v = 0$ always exists. To eliminate cases where there is evidently no unique solution, we look for the cases where there exists an additional solution of the form $(x, y, z, v) = (1, P, -P, -1)$, for some P. Substituting the first equation into the second, we obtain

$$P^2 + 2(\lambda - 1)P + \lambda = 0.$$

This quadratic has real roots if and only if $(\lambda-1)^2 - \lambda \geq 0$, that is, for $\lambda \geq \dfrac{3+\sqrt{5}}{2}$ or $\lambda \leq \dfrac{3-\sqrt{5}}{2}$.

We conclude that for any real λ such that $\lambda \notin [\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}]$, the system *does not* have a unique solution.

We now guess, and try to prove the following statement: for all values of λ such that

$$\frac{3-\sqrt{5}}{2} \leq \lambda \leq \frac{3+\sqrt{5}}{2} \tag{3}$$

the system has only one solution, which is the trivial one.

To prove the statement, note that

$$(1) \Rightarrow x = -(y+z+v) \overset{(2)}{\Rightarrow} y^2 + yv + \lambda yz + \lambda(z+v)^2 - zv = 0$$

which after rearrangement, reads

$$y^2 + (v + \lambda z)y + \lambda(z + v)^2 - zv = 0.$$

This equation has a real solution if and only if its discriminant is nonnegative, that is,

$$\begin{aligned} (v + \lambda z)^2 - 4\big(\lambda(z + v)^2 - zv\big) &\geq 0 \\ \Rightarrow (4\lambda - 1)v^2 + \lambda(4 - \lambda)z^2 + 2(-3\lambda + 2)vz &\leq 0. \qquad (4) \end{aligned}$$

In particular, it follows from (3) that $4\lambda - 1 < 0$. Therefore, if $z = 0$, the inequality (4) can hold only if $v = 0$ as well. In this case the equations of the original system imply that $y = 0$ and $x = 0$, and we therefore return to the case of a unique (trivial) solution. On the other hand, if $z \neq 0$, we can regard the left hand side of the inequality (4) as quadratic in the variable v, with a negative leading coefficient. This quadratic can attain a nonpositive value only if

$$\begin{aligned} 4(-3\lambda + 2)^2 - 4\lambda(4\lambda - 1)(4 - \lambda) &= 16(\lambda^3 - 2\lambda^2 - 2\lambda + 1) \\ &= 16(\lambda + 1)(\lambda^2 - 3\lambda + 1) \\ &\geq 0. \end{aligned}$$

This contradicts (1) and the desired conclusion follows immediately.

1992

1. Denote $d = \sqrt{c}$. We have

$$c^n + c^{-n} - 2 = (d^n - d^{-n})^2,$$
$$c + c^{-1} - 2 = (d - d^{-1})^2.$$

The required inequality is equivalent to

$$\frac{d^n - d^{-n}}{d - d^{-1}} \geq n.$$

Now we have

$$\begin{aligned} \frac{d^n - d^{-n}}{d - d^{-1}} &= d^{1-n}\frac{d^{2n} - 1}{d^2 - 1} \\ &= d^{1-n}(1 + d^2 + d^4 + \ldots + d^{2n-2}) \\ &\geq d^{1-n} n \sqrt[n]{d^{2+4+\cdots+(2n-2)}} \\ &= n. \end{aligned}$$

with which we are done.

2. We choose ten numbers $a_1, a_2, \ldots, a_{10}$ from S, having distinct last digits.

Denote $T = a_1 + a_2 + \cdots + a_{10}$. We define the following set of twenty numbers

$$A = \{a_1, a_2, \ldots, a_{10}, T - a_1, T - a_2, \ldots, T - a_{10}\}$$

and show that its elements have at least 19 distinct residues modulo 2000. We consider the following cases:

a. If $a_i \equiv a_j \pmod{2000}$, for some distinct indices i and j, we obtain a contradiction to the assumption that $a_i \not\equiv a_j \pmod{10}$.

b. Similarly, if $T - a_i \equiv T - a_j \pmod{2000}$, for some distinct indices i and j we obtain a contradiction.

c. Suppose that $T - a_i \equiv a_j \pmod{2000}$. It then follows that $T - a_i - a_j$, which is the sum of eight numbers from S, is divisible by 2000 and our problem is solved. Thus, we assume that this option does not hold.

d. Assume that

$$T - a_i \equiv a_i \pmod{2000},$$

i.e.

$$T \equiv 2a_i \pmod{2000}$$

for some index i. This can occur only for one index i, because if $T \equiv 2a_j \pmod{2000}$ for $i \neq j$ then $a_i - a_j \equiv 0 \pmod{1000}$, in contradiction with our assumption on the last digits of the chosen numbers.

It follows that we can choose 19 numbers from A, $\alpha_1, \alpha_2, \ldots, \alpha_{19}$ having 19 distinct residues modulo 2000.

We now arrange the elements of S in a way that the first ten are $a_1, a_2, \ldots, a_{10}$, and denote the sum of the first i elements in this sorted set by S_i.

If there exist two elements in $S_{11}, S_{12}, \ldots, S_{1992}$ having the same residue modulo 2000, then the difference between the largest and the smallest one is a sum of elements from S and it is divisible by 2000, so the problem is solved.

Recall that each one from the 19 numbers $\alpha_1, \alpha_2, \ldots, \alpha_{19}$ is a sum of elements $a_1, a_, \ldots, a_{10}$, therefore for the same reason it follows that $\alpha_k \not\equiv S_l\,(2000)$ for each $1 \leq k \leq 19$ and $1 \leq l \leq 1992$. Altogether we have $19 + 1982 = 2001$ distinct residues modulo 2000, which is a contradiction, and the required statement is therefore proved.

3. For each $1 \leq r \leq 100$ we have

$$f_r(1) \geq \frac{1}{2} \Rightarrow a_1^{(r)} = 1.$$

Therefore, $a_2^{(r)} = 2$ or $a_2^{(r)} = 3$ for each $1 \leq r \leq 100$. Consequently, there exist distinct $1 \leq r, s \leq 100$, for which $a_2^{(r)} = a_2^{(s)}$ and thus

$$f_{rs}(1) = 1 \geq \frac{8 \cdot 1}{33}, \quad f_{rs}(2) = 1 \text{ or } 2 \geq \frac{8 \cdot 2}{33},$$

$$f_{rs}(3) = 2 \geq \frac{8 \cdot 3}{33}, \quad f_{rs}(4) = 2 \geq \frac{8 \cdot 4}{33}, \quad f_{rs}(5) = 2 \geq \frac{8 \cdot 5}{33}.$$

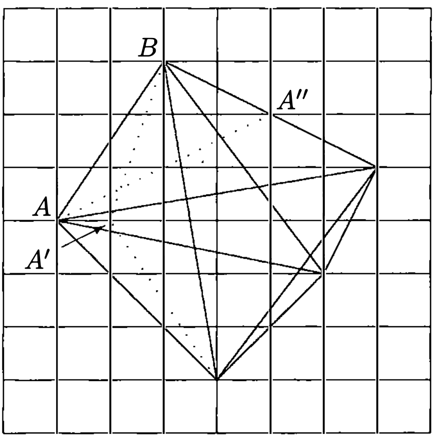
4. For every pentagon P we denote the pentagon formed by intersection points of its five diagonals by $Q(P)$. We also denote the "star" defined by the diagonals of the original pentagon by $R(P)$. Clearly, we have $R(P) \supset Q(P)$.

Assume that there exists a convex pentagon P_0 for which $Q(P_0)$ contains no lattice point. If the pentagon P_0 or its boundary contain five points which define a pentagon P_1 for which $Q(P_1)$ also contains no lattice point, we replace P_0 by P_1.

Since the number of lattice points of P_0 is finite, we can repeat this process until we end up with a pentagon P for which $Q(P)$ contains no lattice point, and such that P contains no sub-pentagon having this property.

We now observe that in every convex pentagon whose vertices are lattice points, one of the midpoints of its sides or diagonals is a lattice point. This follows easily by considering the parity of the coordinates of the vertices.

If $R(P)$ or on its boundary contains a lattice point we consider a pentagon P' defined by replacing the corresponding vertex (say A) by a lattice point in the star (A') (see figure).



Since $Q(P') \subset Q(P)$, we obtain a convex pentagon P' contained in P and $Q(P')$ contains no lattice point. This contradicts the definition of P.

Since P contains at least one lattice point (a midpoint of a side or of a diagonal), and this point is not inside $R(P)$ or on its boundary, it follows that this point is necessarily the midpoint of a side in P.

Now we can consider the (convex) pentagon P'' whose vertices are lattice points, which is obtained by replacing B by A'',

Since $Q(P'') \subset R(P)$ (by convexity), it follows that $Q(P'')$ contains no lattice point and this contradicts the definition of P.

These contradictions result from assuming the existence of the pentagon P_0, and the statement is therefore proved.

1993

1. Assuming that $10^{n-1} \le a < 10^n$ for some positive integer n we obtain

$$\frac{a}{b} = b.a \Leftrightarrow \frac{a}{b} = \frac{10^n b + a}{10^n} \Leftrightarrow \frac{ab}{a - b^2} = 10^n.$$

We now prove that ab and $a - b^2$ are relatively prime. If we assume that they have a common prime factor p, it follows also that $a^3 = a^2(a - b^2) - ab \cdot ab$ and $b^3 = ab - b(a - b^2)$ have the same common prime factor p, and this contradicts the assumption that a and b are relatively prime.

Therefore, it follows that $a - b^2 = 1$ and, consequently, that $ab = 10^n$. Since a and b are relatively prime and $a = 1 + b^2 > b$, we conclude that $b = 2^n$, $a = 5^n$. Therefore

$$a = 1 + b^2 \Rightarrow 2^{2n} + 1 = 5^n \Rightarrow 4^n + 1 = 5^n \Rightarrow 4^n + 1 = (4+1)^n$$

This is possible only if $n = 1$. In this case we have $b = 2$, $a = 5$. Consequently, $\frac{5}{2} = 2.5$ is the only solution.

2. We denote $x - 1 = y$, and obtain $f(y^2 - 1) = [f(y - 1)]^2$. Now, assume that y_0 is a number, not necessarily real, such that $y_0 - 1$ is a root of f.

We denote by y_1 one of the complex square roots of y_0, that is, $y_1^2 = y_0$. It follows that

$$[f(y_1 - 1)]^2 = f(y_1^2 - 1) = f(y_0 - 1) = 0,$$

which implies that $\sqrt{y_0} - 1$ is also a root of f. Similarly, we conclude that $\sqrt{\sqrt{y_0}} - 1$ is a root of f. It is now clear that for every positive integer n, $\sqrt[2^n]{y_0} - 1$ is also a root of f.

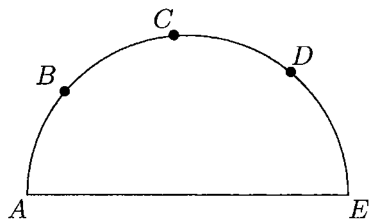
Since the number of roots of f is finite, it follows that $y_0 = 0$, or $y_0 = 1$. Therefore, f has at most two roots, namely (-1) and 0. This implies that

$$f(x) = C(x+1)^k x^m, \qquad k, m \in N$$

for some constants C. Applying this to the identity given in the statement of the problem, we have $m = 0$ and then $C = 0$ or $C = 1$. Consequently, the only two possibilities are $f(x) = (x+1)^k$, $k \in N$, or $f(x) = 0$.

It is now easy to verify directly that these are indeed valid solutions.

3. We may assume that A and E are the ends of the diameter of the semicircle. If this is not the case, we can map A and E to be the ends of the diameter, and the lengths of AB and DE increase. Thus, the left hand side of the desired inequality increases, and the inequality becomes even stronger.



Applying the cosine law three times, we conclude that the required inequality

$$AB^2 + BC^2 + CD^2 + DE^2 + AB \cdot BC \cdot CD + BC \cdot CD \cdot DE < 4$$

implies that

$$\begin{aligned} & AB \cdot BC \cdot CD + BC \cdot CD \cdot DE \\ < \;& -2AB \cdot BC \cdot \cos \angle ABC - 2DE \cdot CD \cdot \cos \angle CDE. \end{aligned}$$

To prove the last inequality, it is sufficient to show that

$$\begin{aligned} & AB \cdot BC \cdot CD < -2AB \cdot BC \cdot \cos \angle ABC, \\ \text{and} \quad & BC \cdot CD \cdot DE < -2CD \cdot DE \cdot \cos \angle CDE. \end{aligned}$$

By symmetry, it is sufficient to prove only one of the last two inequalities, say

$$CD < -2 \cos \angle ABC. \tag{1}$$

Since

$$90^\circ - \frac{CE}{2} = \frac{180^\circ - CE}{2} = \frac{AC}{2} = 180^\circ - \angle ABC$$

and

$$CD = 2 \sin \frac{CD}{2}$$

the inequality (1) can be written as

$$\sin \frac{CD}{2} < \sin \frac{CE}{2}. \tag{2}$$

Since

$$0 < CD < CE < 180^\circ,$$

It follows that

$$0 < \frac{CD}{2} < \frac{CE}{2} < 90^\circ$$

Finally, since the sine function is increasing on the interval $[0, 90^\circ]$, the inequality (2) follows, and this completes the proof.

4. We divide the board S into n^2 sub-squares of size 3×3. We place four rooks in all of the squares that intersect the main diagonal, in the following way: if the nine subsquares of the square are denoted by a_{ij}, $1 \leq i \leq 3$, the four rooks are placed at $a_{11}, a_{12}, a_{23}, a_{33}$. With this construction we have placed on the board $4n$ rooks that satisfy the requirements of the problem,

We now prove that no more than $4n$ rooks may be placed on the board.

We divide the rooks into three types: rooks which take no other rook, those taking other rooks horizontally, and those taking other rooks vertically.

Assume that there are k rooks of the first kind, l rooks of the second kind, and m rooks of the third kind. In any good arrangement the k rooks of the first kind "occupy" k rows and k columns, l rooks of the second kind occupy $\frac{l}{2}$ rows, and l columns, and m rooks of the third kind occupy m columns and $\frac{m}{2}$ rows. Altogether, the number of occupied rows and columns is

$$2k + \frac{3}{2}l + \frac{3}{2}m.$$

We now recall that there exist $6n$ rows and columns, and therefore, conclude that

$$6n \geq 2k + \frac{3}{2}l + \frac{3}{2}m \geq \frac{3}{2}(k + l + m).$$

Finally, this leads to $k + l + m \leq 4n$.

1994

1. We assume that $n > m$. Consider the left condition (which refers to n). The x-s which satisfy it lie in the "good" interval of the form

$$\left[\frac{k}{n} + \frac{1}{3n}, \frac{k}{n} + \frac{2}{3n}\right]$$

(k is an integer) whose length is $\frac{1}{3n}$.

These are separated by open "bad" intervals of the form

$$\left(\frac{k}{n} - \frac{1}{3n}, \frac{k}{n} + \frac{1}{3n}\right)$$

(k is an integer) of length $\frac{2}{3n}$ where the condition does not hold. We call them n-good and n-bad intervals. Similarly, there are m-good and m-bad intervals.

We now assume that no x satisfies

$$1/3 \leq \{xn\} \leq 2/3$$

and

$$1/3 \leq \{xm\} \leq 2/3.$$

Then every m-good interval is contained in one of the n-bad intervals. Since an n-bad interval is an open interval of length $\frac{2}{3n}$, whereas an m-good interval is a closed interval of length $\frac{1}{3m}$, we conclude that

$$\frac{1}{3m} < \frac{2}{3n}$$

and consequently that $n < 2m$.

Now since $n < 2m$ we see that

$$\frac{2}{3n} > \frac{1}{3m},$$

which means that n-good interval

$$\left[\frac{1}{3n}, \frac{2}{3n}\right]$$

has common points with m-good interval

$$\left[\frac{1}{3m}, \frac{2}{3m}\right].$$

Hence for any
$$\frac{2}{3n} \leq x \leq \frac{1}{3m},$$
we have
$$1/3 \leq \{xn\} \leq 2/3$$
and
$$1/3 \leq \{xm\} \leq 2/3.$$

Remark: The statement is not true if the condition $1/3 \leq \{xn\} \leq 2/3$ and $1/3 \leq \{xm\} \leq 2/3$ is replaced by $1/3 < \{xn\} < 2/3$ and $1/3 < \{xm\} < 2/3$. A counterexample is for $n = 2$, $m = 1$.

2. Define
$$x = a_1 + \cdots + a_k, \qquad y = \frac{1}{a_1} + \cdots + \frac{1}{a_k},$$
$$a = a_{k+1} + \cdots + a_n, \qquad b = \frac{1}{a_{k+1}} + \cdots + \frac{1}{a_n}.$$
We note that
$$n + \sum_{i,j,\ i \neq j} \frac{a_i}{a_j} = \sum_{i,j} \frac{a_i}{a_j} = (x+a)(y+b)$$
Using the Cauchy-Schwartz inequality twice we obtain
$$(x+a)(y+b) \geq (\sqrt{xy} + \sqrt{ab})^2 \geq (k + \sqrt{ab})^2.$$
The minimum of $(x+a)(y+b)$ is obtained by taking
$$a_1 = a_2 = \ldots = a_k = \sqrt{ab}.$$

3. Denote the centres of the three circles by A'', B'', C''. P is the circumcentre of the triangle $A''B''C''$. Since the circles have the same radii, it follows that the segment joining the midpoint of PB and the midpoint of PC joins also the midpoint of $A''C''$ and the midpoint of $A''B''$. This segment is therefore equal (and parallel) to $\frac{1}{2}BC$ and to $\frac{1}{2}B''C''$.

An analogous result is obtained for the other two sides, and it follows that the triangles ABC and $A''B''C''$ are congruent. The sides of triangle $A'B'C'$ are parallel to those of $\triangle A''B''C''$ and they are therefore similar to $\triangle ABC$.

Moreover, the centre of the incircle of $\triangle A''B''C''$ is also the centre of the incircle of $\triangle A'B'C'$, and therefore the similarity ratio between these triangles is $(R+r) : r$ where r is the inradius of $\triangle A''B''C''$ and R is the circumradius which is also the circumradius of $\triangle A''B''C''$.

Using the well known inequality between the inradius and the circumradius of a triangle, we have $R \geq 2r$. Therefore, the ratio r at least 3 and therefore the ratio of the areas is at least 9.

4. Take $n_0 = 9$, $m_0 > 4 \cdot 2^9$. Assume that an m_0-n_0-society is given and let B be the set of 9 girls. For each of the m_0 boys consider the set of girls that he knows. Since there are only 2^9 distinct sets of girls and more than $4 \cdot 2^9$ boys, there exists a set A of 5 boys who know exactly the same set C of girls.

If C contains 5 or more girls, we are done.

If C contains less than 5 girls, then the complementary set contains at least 5 girls that no boy from A knows and we are done again.

1995

1. We first note that if $\sqrt{a}-\sqrt{b}>1$, that is, $a-b>1+2\sqrt{b}$, then there is a square between the integers a and b.

We now denote the n-th prime by p_n. It is now enough to show that

$$p_{n+1}=S_{n+1}-S_n>1+2\sqrt{S_n}.$$

For $n\leq 4$, we have

$$S_1=2,\quad S_2=5,\quad S_3=10,\quad S_4=17,\quad S_5=28.$$

and it is easy to verify the statement directly. Suppose now that $n\geq 5$, and write $P_n=2k-1$, i.e., $k=(1/2)(P_n+1)$. Then,

$$\begin{aligned} S_n &= 2+3+5+7+11+\cdots+P_n \\ &< 1+3+5+7+9+\cdots+(P_n-2)+P_n \\ &= k^2, \end{aligned}$$

so

$$2\sqrt{S_n}-1<2k-1=P_n.$$

We now have $1+2\sqrt{S_n}<P_n+2\leq P_{n+1}$ and we are done.

2. For any triangle ABC, we shall denote its area by S_{ABC}. With no loss of generality, we assume that the radius of the circle is $R=1$. Using the fact that the area T of a triangle whose sides are a, b, c and circumradius R is $T=abc/4R$, we have

$$2S_{PP_1P_2}=d_{12}\cdot P_1P_2=\frac{1}{2}PP_1\cdot PP_2\cdot P_1P_2,$$

and consequently,

$$d_{12}=\frac{1}{2}PP_1\ PP_2.$$

.

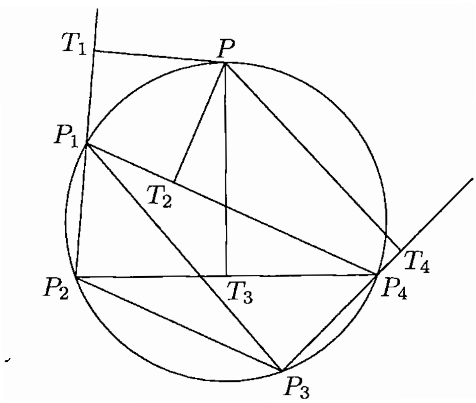
Similarly, we obtain

$$\begin{aligned} d_{34} &= \frac{1}{2} PP_3\, PP_4, \\ d_{13} &= \frac{1}{2} PP_1\, PP_3, \\ \text{and}\quad d_{24} &= \frac{1}{2} PP_2\, PP_4. \end{aligned}$$

and the result follows.

Alternative solution:

The two angles $\angle P_1P_2P$, $\angle P_1P_3P$ (see the figure) are equal as they lie on the same arc.



This implies the similarity

$$\triangle PT_1P_2 \sim \triangle PT_2P_3, \quad \triangle PT_4P_3 \sim \triangle PT_3P_2$$

where T_1, T_2, T_3, T_4 are the feet of the perpendiculars from P onto P_2P_1, P_1P_3, P_2P_4, P_3P_4, respectively. We now have

$$\frac{d_{12}}{d_{13}} = \frac{PT_1}{PT_2} = \frac{PP_2}{PP_3}$$

$$\frac{d_{24}}{d_{34}} = \frac{PT_3}{PT_4} = \frac{PP_2}{PP_3}$$

and therefore $d_{12}d_{34} = d_{13}d_{24}$.

Note here that this proof is not yet complete, as it depends on the cyclic order of the points P, P_1, P_2, P_3, P_4 as it appears in the figure. For example, $\angle P_1P_2P$ and $\angle P_1P_3P$ are not equal if P_3 lies between P and P_1 on the arc opposite to the one in the figure. In this case $\angle P_1P_2P + \angle P_1P_3P = 180°$. To complete the proof, we simply note that if we replace P_1 by P_3 and replace d_{12} with d_{13} appropriately, then the claim of the problem remains unchanged.

3. We check the values of $f(x)$ at 0, $\frac{1}{2}, 1$. This gives

$$|c| \leq 1, \quad |a+2b+4c| \leq 4, \quad |a+b+c| \leq 1.$$

We denote

$$A = a + 2b + 4c, \qquad B = a + b + c.$$

This implies

$$a = -A + 2B + c, \qquad b = A - B - 3c.$$

It follows that

$$|a| \leq |A| + 2|B| + 2|c| \leq 4 + 2 + 2 = 8$$

and

$$|b| \leq |A| + |B| + 3|c| \leq 4 + 1 + 3 = 8$$

so

$$|a| + |b| + |c| \leq 8 + 8 + 1 = 17.$$

This yields an upper bound for $|a| + |b| + |c|$.

Finally, the example $f(x) = 8x^2 - 8x + 1$ shows that the value 17 can indeed be attained, and the result follows.

4. Consider a convex triangular polyhedron with $n \geq 4$ vertices. We choose an arbitrary vertex C. Suppose that $r \geq 3$ edges emanate from C. Due to the polyhedral structure, there exists a proper labeling of these neighboring vertices, say, $CC_1, CC_2, \ldots, CC_r$ in a way that CC_1, C_1C_2, C_2C_3, ..., $C_{r-1}C_r$, C_rC_1 are edges. We colour CC_1 in blue, $CC_2, CC_3, \ldots, CC_r$ in red, C_1C_2 in red, and $C_2C_3, C_3C_4, \ldots, C_rC_1$ in blue.

We now proceed by rounds: we choose any triangular face F which has either one or two coloured edges. We consider the following two cases:

(a) If F has only one coloured edge, we colour the remaining two edges with two opposite colours: one in red and one in blue. Note that before this colouring, F had precisely one vertex which was yet unreachable by coloured paths. We call it a "new vertex".

(b) If F has two coloured edges, we colour the third edge with any colour. Note that all of the vertices of F were already reachable by coloured paths even before the colouring of this third edge.

From our colouring it follows that in case (a) we can reach the new vertex by both red and blue paths. In case (b) no new vertex is reached. At the end of this round, the vertices that belong to completely coloured faces can be reached from C by both red and blue paths.

We continue with these rounds inductively: at each step we seek a new face with either one or two coloured edges and repeat (a) or (b), accordingly. This procedure preserves the good colouring property of the subset of vertices that belong to completely coloured faces. We conclude after all edges have been coloured. This must occur within a finite number of steps. Otherwise, we would have two *disjoint* subsets of faces: one which is completely coloured and the other which is completely uncoloured. This contradicts the polyhedral structure. At that point we are done.

1996

1. The only solution is the trivial one, namely $(0, 0, \dots, 0)$.

Let $m = 1997$. If

$$(x_1, x_2, \dots, x_m)$$

solves (1) then so does

$$\frac{1}{2}(x_1, x_2, \dots, x_m).$$

Therefore, we assume that one of the terms x_k is odd. Let the first odd x_k be x_{q+1} where $q \in 0, 1, 2, \dots, m-1$, which implies that $x_k = 2u_k$ for some integer u_k for any $k \leq q$. Now, it follows that the left hand side of (1) equals $2^q \times$ an odd number, and the right hand side of (1) equals $2^q \times$ (an even number) and this is a contradiction.

2. If b is a positive integer then

$$(b+1)^3 - b^3 = 3b^2 + 3b + 1 = n^2,$$

that is

$$12b^2 + 12b + 3 = 4n^2 - 1,$$

or

$$3(2b+1)^2 = (2n+1)(2n-1).$$

Since $2n+1$ and $2n-1$ are relatively prime, there are two cases to check:

A) $2n - 1 = 3c^2, 2n + 1 = d^2$. This yields $d^2 - 3c^2 = 2$, which is impossible if we consider modulo 3.

B) $2n - 1 = d^2, 2n + 1 = 3c^2$. Thus d is odd. Then

$$d = 2s + 1,$$

i.e.

$$2n = d^2 + 1 = 4s^2 + 4s + 2 = 2\left((s+1)^2 + s^2\right),$$

or

$$n = (s+1)^2 + s^2,$$

and we are done.

To find an example, we use the first equation. This gives

$$b = \frac{-3 + \sqrt{12n^2 - 3}}{6}.$$

The first $n > 2$ for which the discriminant is a square is $n = 13$, giving $b = 7$. Indeed,

$$8^3 - 7^3 = 512 - 343 = 13^2$$

and

$$13 = 2^2 + 3^2.$$

Remark: Using Pell's equations with the discriminant $12n^2 - 3$ we can get all the solutions. The next solution is $n = 181$, noting

$$105^3 - 104^3 = 181^2, \quad \text{where} \quad 181 = 9^2 + 10^2.$$

3. Denote the number of vertices, edges and faces by v, e, f, respectively, the number of vertices of degree i by v_i, the number of faces with i edges by f_i. We have:

$$\begin{aligned} v &= (v_3 = 0) + v_4 + v_5 + v_6 + \ldots \\ f &= f_3 + f_4 + f_5 + f_6 + \ldots \\ 2e &= (3v_3 = 0) + 4v_4 + 5v_5 + 6v_6 + \ldots \\ 2e &= 3f_3 + 4f_4 + 5f_5 + 6f_6 + \ldots \end{aligned}$$

Now by Euler's formula,

$$4v + 4f = 4e + 8 = 2e + 2e + 8,$$

so

$$\begin{aligned} & 4(v_4 + v_5 + v_6 + \ldots) + 4(f_3 + f_4 + f_5 + \ldots) \\ = \ & (4v_4 + 5v_5 + \ldots) + (3f_3 + 4f_4 + 5f_5 + \ldots) + 8, \end{aligned}$$

i.e.

$$f_3 = 8 + (v_5 + 2v_6 + \ldots) + (f_5 + 2f_6 + \ldots) \geq 8$$

4. We denote $S = a_1b_1 + \ldots + a_nb_n$ and write S as:

$$\begin{aligned} S = \ & a_1(b_1 - b_2) + (a_1 + a_2)(b_2 - b_3) \\ & +(a_1 + a_2 + a_3)(b_3 - b_4) \\ & \ldots + (a_1 + a_2 + \ldots + a_{n-1})(b_{n-1} - b_n) \\ & +(a_1 + \ldots + a_n)b_n. \end{aligned}$$

Let $A_i = |a_1 + a_2 + \ldots + a_i|$ and suppose that A_k is maximal among the A_is. Then,

$$|S| \leq \sum_1^n A_i(b_i - b_{i+1}) \leq A_k \sum_1^n (b_i - b_{i+1}) = A_k b_1 \leq A_k$$

(by definition, we assume $b_{n+1} = 0$). This completes the proof.

1997

1. The answer is positive in the following general context. Suppose that x, n are positive integers and n is even. Then there exists a positive integer N such that

$$(\sqrt{x+1}-\sqrt{x})^n = \sqrt{N}-\sqrt{N-1}.$$

Proof: Note that

$$\begin{aligned}
&\left(\sqrt{x+1}-\sqrt{x}\right)^n \\
= &\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\sqrt{x+1})^k(\sqrt{x})^{n-k} \\
= &\sum_{k=0}^{n/2}\binom{n}{2k}(x+1)^k x^{\frac{n-2k}{2}} \\
&\quad -\frac{\sqrt{x+1}}{\sqrt{x}}\times\sum_{k=0}^{n/2-1}\binom{n}{2k+1}(x+1)^k x^{\frac{n-2k}{2}}. \qquad (1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(\sqrt{x+1}+\sqrt{x}\right)^n \\
= &\sum_{k=0}^{n}\binom{n}{k}(\sqrt{x+1})^k(\sqrt{x})^{n-k} \\
= &\sum_{k=0}^{n/2}\binom{n}{2k}(x+1)^k x^{\frac{n-2k}{2}} \\
&\quad +\frac{\sqrt{x+1}}{\sqrt{x}}\times\sum_{k=0}^{n/2-1}\binom{n}{2k+1}(x+1)^k x^{\frac{n-2k}{2}}. \qquad (2)
\end{aligned}$$

We denote the sum

$$\sum_{k=0}^{n/2}\binom{n}{2k}(x+1)^k x^{\frac{n-2k}{2}}$$

by S_1 and the sum

$$\frac{\sqrt{x+1}}{\sqrt{x}} \cdot \sum_{k=0}^{n/2-1} \binom{n}{2k+1} (x+1)^k x^{\frac{n-2k}{2}}$$

by S_2. Multiplying (1) by (2), we end up with

$$\begin{aligned} 1 &= \left(\sqrt{x+1}+\sqrt{x}\right)^n \times \left(\sqrt{x+1}-\sqrt{x}\right)^n \\ &= (S_1 - S_2) \times (S_1 + S_2) \\ &= S_1^2 - S_2^2. \end{aligned}$$

Note that S_1 is an integer and therefore $N = S_1^2$ is also an integer. Consequently, we get

$$\left(\sqrt{x+1}-\sqrt{x}\right)^n = S_1 - S_2 = \sqrt{N} - \sqrt{N-1}.$$

as required.

2. The set of numbers α that satisfy the required condition is the set of all integers.

Suppose that α is an integer. Then, for each n we may choose $m = \alpha n$, and the condition is satisfied.

Conversely, suppose that the condition holds for some α. Let m_k be the corresponding numerators for $n = 2^k$, $k = 0, 1, \ldots$.

If

$$\frac{m_k}{2^k} \neq \frac{m_{k+1}}{2^{k+1}},$$

then

$$\left|\frac{m_k}{2^k} - \frac{m_{k+1}}{2^{k+1}}\right| = \left|\frac{2m_k - m_{k+1}}{2^{k+1}}\right| \geq \frac{1}{2^{k+1}}.$$

Therefore, using the triangle inequality,

$$\begin{aligned} \frac{1}{2^{k+1}} &\leq \left|\frac{m_k}{2^k} - \frac{m_{k+1}}{2^{k+1}}\right| \\ &\leq \left|\frac{m_k}{2^k} - \alpha\right| + \left|\alpha - \frac{m_{k+1}}{2^{k+1}}\right| \\ &< \frac{1}{3 \cdot 2^k} + \frac{1}{3 \cdot 2^{k+1}} \\ &= \frac{1}{2^{k+1}}, \end{aligned}$$

and this yields a contradiction. It follows that $m_k/2^k = m_{k+1}/2^{k+1}$ for all k, and thus all the approximations of the form $m_k/2^k$ to α are equal to an integer m_0. But then

$$|\alpha - m_0| = \left|\alpha - \frac{m_k}{2^k}\right| \le \frac{1}{3 \cdot 2^k}$$

for every k. Consequently $\alpha - m_0 = 0$ and α is therefore an integer.

3. Denote $AB = c$, $AC = b$, $BC = a$ and $OA_1 = a_1$, $OB_1 = b_1$, $OC_1 = c_1$.

We start with the following statement.

Statement:

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} = 1. \tag{1}$$

Proof: Denote $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle BCA$. Denote also $\delta = 90° + \gamma - \beta$. We note that

$$\angle OBC = 90° - \alpha.$$

Then by the sine law, applied to $\triangle OA_1B$, we have

$$\frac{OA_1}{\sin(90° - \alpha)} = \frac{OA_1}{\cos\alpha} = \frac{R}{\sin\delta}$$

and in $\triangle AA_1B$ we have

$$\frac{AA_1}{\sin\beta} = \frac{c}{\sin\delta} = \frac{2R\sin\gamma}{\sin\delta}.$$

Hence,

$$\frac{OA_1}{AA_1} = \frac{\cos\alpha}{2\sin\beta\sin\gamma}.$$

Similarly, we have

$$\frac{OB_1}{BB_1} = \frac{\cos\beta}{2\sin\alpha\sin\gamma}, \qquad \frac{OC_1}{CC_1} = \frac{\cos\gamma}{2\sin\alpha\sin\beta}.$$

Addition with common denominator $2\sin\alpha\sin\beta\sin\gamma$ yields

$$\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1}$$

$$= \frac{\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma}{2\sin\alpha\sin\beta\sin\gamma}.$$

It now remains to show that

$$\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma=2\sin\alpha\sin\beta\sin\gamma. \tag{2}$$

This latter equality follows from considering areas. We denote the area of $\triangle XYZ$ by S_{XYZ}. Then

$$S_{ABC}=8p^2\sin\alpha\sin\beta\sin\gamma.$$

The angles of $\triangle ABC$ are arcs on the circumcircle, and therefore

$$\angle BOC=2\alpha,\quad \angle COA=2\beta,\quad \angle AOB=2\gamma.$$

Now,

$$S_{BOC}=2p^2\sin 2\alpha=4p^2\sin\alpha\cos\alpha,$$

and similarly,

$$S_{COA}=4p^2\sin\beta\cos\beta,\qquad S_{AOB}=4p^2\sin\gamma\cos\gamma.$$

From the equality $S_{ABC}=S_{BOC}+S_{COA}+S_{AOB}$ it follows that

$$8p^2\sin\alpha\sin\beta\sin\gamma=4p^2(\sin\alpha\cos\alpha+\sin\beta\cos\beta+\sin\gamma\cos\gamma)$$

and (2) is obtained through division by $4p^2$.
When (1) is written in terms of a_1, b_1, c_1, p, we obtain

$$\frac{a_1}{2p+a_1}+\frac{b_1}{2p+b_1}+\frac{c_1}{2p+c_1}=1.$$

This yields

$$4p^3-p(a_1b_1+b_1c_1+c_1a_1)=a_1b_1c_1. \tag{3}$$

From the assumptions, it follows that $p|a_1b_1c_1$. Since p is prime, it follows that it divides at least one of a_1, b_1 or c_1. We assume, with no loss of generality, that $p|a_1$. Since the diameter is the longest chord in a circle, we have $a_1<2p$, and therefore $a_1=p$. Equation (3) becomes

$$4p^2-p(b_1+c_1)=2b_1c_1. \tag{4}$$

Now, $p|2b_1c_1$. If $p > 2$, it follows that, for instance, $p|b_1$, and again, since $b_1 < 2p$, we obtain $p = b_1$. This lead to $p = c_1$ and to the conclusion that the triangle is equilateral.

If $p = 2$ it follows from (4) that

$$(b_1 + 1)(c_1 + 1) = 9,$$

i.e., $b_1 = c_1 = 2$. Now it also follows that $a_1 = 2$ and the triangle is equilateral.

Finally, the only solution is an equilateral triangle with $a = b = c = 2P\sqrt{3}$.

4. **Solution 1**

We start constructing a sequence by first choosing a occurrences of A for some odd a. Since B and C must appear at least once, we have $1 \le a \le 1995$. The number of such choices is $\binom{1997}{a}$. For any choice of A's we can choose b occurrences of B for some odd b where $1 \le b \le 1996 - a$. We have $\binom{1997-a}{b}$ choices. Choosing a and b already determines the sequence. Therefore, the required number of sequences is

$$S = \sum_{a=1, a\ odd}^{1995} \binom{1997}{a} \times \sum_{b=1, b\ odd}^{1996-a} \binom{1997-a}{b}$$

Since the sum

$$\sum_{i=1, i\ odd}^{N} \binom{N}{i} = \frac{1}{2} \cdot 2^N$$

We have

$$S = \frac{1}{2} \sum_{a=1, a\ odd}^{1995} \binom{1997}{a} 2^{1997-a}$$

We now use the expansions

$$\begin{aligned} (1+x)^N &= \sum_{i=1}^{N} \binom{N}{i} x^{N-i} \\ \text{and} \quad (1-x)^N &= \sum_{i=1}^{N} \binom{N}{i} (-1)^{N-i} x^{N-i}, \end{aligned}$$

and write

$$(1+x)^N - (1-x)^N = 2 \sum_{i=1, i \text{ odd}}^{N} \binom{N}{i} x^{N-i}.$$

In our case, with $N = 1997$ and $x = 2$ we finally have

$$S = \frac{1}{2}\left(\frac{1}{2}(3^{1997} - 1) - 1\right) = \frac{1}{4}(3^{1997} - 3).$$

Solution 2

For every odd $n \geq 3$ we denote the number of sequences of length n, composed of the letters A, B, C and such that each letter occurs an odd number of times, by x_n. We use y_n to denote the number of sequences of length n composed of the letters which can be A, B, C such that the number of occurences of A, B, C is odd, even, even, respectively. Similarly, we use z_n to denote the number of sequences of length n with even, odd, even occurences of A, B, C, respectively. Finally, let w_n denote the number of such with even, even, odd, occurences of A, B, C, respectively.

By symmetry we have $y_n = z_n = w_n$. Since n is odd n the sequences counted by x_n, y_n, z_n, and w_n cover all possible sequences of length n that are composed of the letters A, B, C. We therefore conclude that $x_n + 3y_n = 3^n$.

For $n = 3$ it is easy to see that $x_n = 6$ and $y_n = z_n = w_n = 7$ (zero number of occurences is counted as even).

We now examine x_{n+2}. Sequences of length $n + 2$ can be obtained from a sequence of the type x_n by adding AA, BB or CC. They can be obtained from sequence of the type y_n by adding AB or BA, from z_n by adding AC or CA, and from w_n by adding BC or CB. We therefore have

$$x_{n+2} = 3x_n + 6y_n$$

or

$$x_{n+2} = x_n + 2 \cdot 3^n.$$

To solve the recurrence relation, we seek a solution of the form $x_n = \alpha \cdot 3^n + \beta$. Tis leads to

$$x_n = \frac{1}{4}(3^n - 3).$$

The required answer is obtained by substituting $n = 1997$ in this formula.

Solution 3

We use generating functions. Suppose that the sequence is composed of a As, b Bs, and c Cs where a, b, c are odd. The number of possibilities for constructing such a sequence is

$$\frac{1997!}{a!\ b!\ c!}$$

where $a+b+c = 1997$. Therefore, we are looking for the sum

$$\sum_{a+b+c=1997,\ a,b,c\ odd} \frac{1997!}{a!\ b!\ c!}.$$

To that end, we note that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \frac{1}{2}(e^x - e^{-x}) = \sum_{k=0,\ k\ odd}^{\infty} \frac{x^k}{k!}.$$

We now use the generating function

$$\begin{aligned} f(x) &= \frac{1}{2}(e^x - e^{-x}) \times \frac{1}{2}(e^x - e^{-x}) \times \frac{1}{2}(e^x - e^{-x}) \\ &= \frac{1}{8}(e^x - e^{-x})^3 \\ &= \frac{1}{8}\left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right) \\ &= \frac{1}{8}\left(\sum_{i=0}^{\infty} \frac{(3x)^i}{i!} - 3\sum_{j=0}^{\infty} \frac{x^j}{j!} + \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} - \sum_{k=0}^{\infty} \frac{(-3x)^k}{k!}\right) \end{aligned}$$

The coefficient of x^{1997} in the latter expansion is

$$\frac{1}{4 \times 1997!}(3^{1997} - 3).$$

On the other hand,

$$f(x) = \sum_{a=0,\ a\ odd}^{\infty} \frac{x^a}{a!} \times \sum_{b=0,\ b\ odd}^{\infty} \frac{x^b}{b!} \times \sum_{k=0,\ c\ odd}^{\infty} \frac{x^c}{c!}.$$

Therefore, the coefficient of x^{1997} in this expansion is

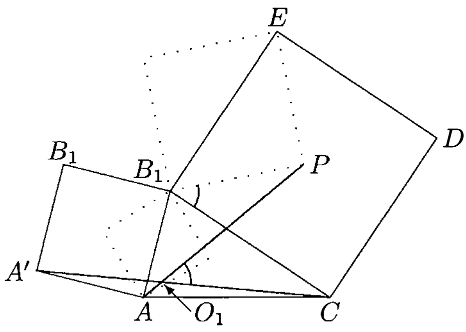
$$\sum_{a+b+c=1997,\ a,b,c\ odd} \frac{1997!}{a!\ b!\ c!}.$$

Finally, it follows that

$$\sum_{a+b+c=1997,\ a,b,c\ odd} \frac{1997!}{a!\ b!\ c!} = \frac{1}{4}(3^{1997} - 3).$$

5. We consider the squares ABB_1A' and $BCDE$ and apply the following transformation: rotation around B by $45°$ and shrinking by the factor $\sqrt{2}$. The point A' is mapped to A and the point C is mapped to P (see figure). The segment $A'C$ is mapped to the segment AP.

It follows that the angle between these two segments is $45°$. Denote the intersection point of these segments by O_1. The angles $\angle CO_1P$ and $\angle CBP$ are equal (both equal $45°$), and the quadrilateral BO_1CP is therefore cyclic. Consequently, the circumcircle of $\triangle BCP$ intersects the line AP at O_1.



Consider now the intersection point O_2 of AP and BA''. By the same considerations, the circle circumscribing $\triangle BCP$ intersects the line AP at O_2. Consequently the points O_1 and O_2 coincide, i.e. the lines AP, $A'C$ and $A''B$ are concurrent.

6. Solution 1

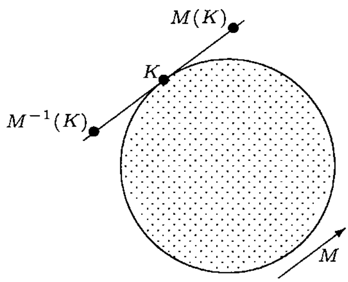
The answer is negative. Suppose that the disk C can be decomposed into the two congruent parts P_1 and P_2. Let f be the congruence mapping that maps P_1 to P_2 and let g be its inverse. Denote the image of the whole disk C under f by C_1 (which is a disk), and

the image of C under g by C_2. The three disks C, C_1 and C_2 are congruent and the union of C_1 and C_2 covers C. It is easy to see now that the two image disks C_1 and C_2 must each cover C. If this is the case, then the three disks must coincide. Since f and g are congruences they must keep the centre of C fixed. Since P_1 and P_2 are disjoint, this yields a contradiction.

Solution 2

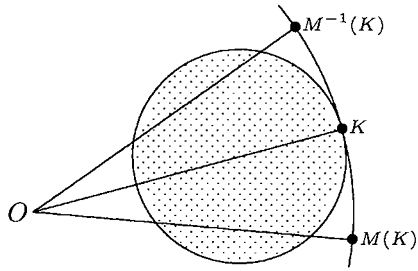
Suppose that we divide the disk into two disjoint congruent parts $\mathcal{A}$ and $\mathcal{B}$. This implies that there exists some isometry transformation M which maps $\mathcal{A}$ onto $\mathcal{B}$. We now consider some properties of isometries: if an isometry in a plane preserves direction then it is a pure translation or a pure rotation. If it reverses direction, then it is a combination of reflection and translation. We investigate and obtain a contradiction in all possibilities.

Case (a). M is a translation and $M(\mathcal{A}) = \mathcal{B}$. Let l be a line tangent to a circle whose direction is parallel to the direction of translation and let K its tangency point to the circle.



There are two possibilities: K belongs to $\mathcal{A}$ or K belongs to $\mathcal{B}$. If K belongs to $\mathcal{A}$ then $M(K) \in \mathcal{B}$. But $M(K)$ is totally outside the circle. If K belongs to $\mathcal{B}$, then $M^{-1}(K) \in \mathcal{A}$ and again, $M^{-1}(K)$ is outside the circle. The contradiction proves that M is not a translation.

Case (b). M is a rotation around the point O. If O is not the centre of the circle, consider the point K at the largest distance from O. As in case (a), $M(K)$ and $M^{-1}(K)$ do not belong to the circle and we thus arrive at a contradiction.



If M is a rotation around the centre of the circle O, then the centre of the circle is a fixed point of M, i.e. if $O \in \mathcal{A}$ then $O = M(O) \in \mathcal{B}$, in contradiction to $\mathcal{A}$ and $\mathcal{B}$ being disjoint. Conversely, if $O \in \mathcal{B}$ then $O = M^{-1}(O) \in \mathcal{A}$, again a contradiction.

Case (c). M is a reflection with respect to a line l. Since all the points of l are fixed points of M, and by considerations mentioned in case (b) there may be no fixed point inside the circle, the line l does not intersect the circle. But then M maps the circle into a circle which is disjoint to it, in contradiction with $M(\mathcal{A}) = \mathcal{B}$.

Case (d). M is a combination of reflection with respect to a line l followed by a translation. If the direction of the translation is perpendicular to l, the M can be described as a reflection with respect to a line l' parallel to l and we are back in case (c). Generally, the translation can be decomposed into two components: one perpendicular to l and the other parallel to l. Therefore M can be described as a combination of reflection with respect to a line l' (parallel to l) followed by a translation at a direction parallel to l'. Consider now the point K of the circle whose distance from l' is maximal (for this consideration it does not matter whether the line l' intersects the circle or not). The points $M(K)$ and $M^{-1}(K)$ do not belong to the circle, and thus we arrive at a contradiction as in cases (a) and (b).

1998

1. I. We denote a success by 1, failure by 0, and the probability that the game ends exactly after n turns by p_n. Every sequence of n turns (coin flips) consists of n steps, each bears the probability $\frac{1}{2}$ to be a 1 or a 0. We therefore have

$$p_n = \frac{1}{2^n} M_n$$

where M_n is the number of n digits binary sequences that end with two consecutive 1s and contains no prior two consecutive 1s. Any such "legal" sequence belongs to one of the following types

(a) It begins with 0, and followed by $n-1$ digits, where the last two digits are 11

(b) It begins with 1, followed by $n-2$ digits where the last two digits are 11.

Consequently, for any $n \geq 5$ we can write the following recurrence relation:

$$M_n = M_{n-1} + M_{n-2}.$$

It is easy to verify that $M_2 = 1$ (the sequence is 11), $M_3 = 1$ (the sequence is 011), and $M_4 = 2$ (the two possible sequences are 1011 and 0011).

Let F_n denote the n-th term of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. It follows that

$$p_n = \frac{1}{2^n} F_{n-1}, \quad n = 2, 3, \ldots.$$

II. We denote the average (expected) number of coin flips till the game terminates by x. We first prove that x is finite, a fact which is not apriori obvious. To that end, we use Binet's formula

$$F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n < \left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Now, the average length of a game is, by definition,

$$x = \sum_{n=1}^{\infty} np_n < \sum_{n=1}^{\infty} \frac{n}{2^n} \left(\frac{1+\sqrt{5}}{2} \right)^{n-1} = \frac{1}{2} \sum_{n=1}^{\infty} n \left(\frac{1+\sqrt{5}}{4} \right)^{n-1}$$

The ratio, r_n, of two consecutive terms in the latter series is

$$r_n = \frac{n\left(\frac{1+\sqrt{5}}{4}\right)^{n-1}}{(n-1)\left(\frac{1+\sqrt{5}}{4}\right)^{n-2}} = \frac{1+\sqrt{5}}{4} \cdot \frac{n}{n-1}.$$

It is easy to see that $r_n < 1$ for $n \geq 7$. Consequently, our series is dominated by a convergent geometric series, and is therefore also convergent.

Knowing that x is finite, we can write equations that involve x as a variable. We consider the following three cases:

(a) the game starts with one 0 (i.e. failure),

(b) the game starts with 10,

(c) the game starts with 11 (and ends).

The respective average game lengths in these three cases are:

$$\frac{1}{2}(x+1), \qquad \frac{1}{4}(x+2) \qquad \text{and} \qquad \frac{1}{2} = 2 \cdot \frac{1}{4}.$$

Therefore, we can write

$$x = \frac{1}{2} + \frac{1}{2}(x+1) + \frac{1}{4}(x+2)$$

and the solution is $x = 6$.

Alternative solution for II: We use the identity

$$S = \sum_{n=1}^{\infty} \frac{1}{2^n} F_n = 2. \tag{1}$$

We write

$$\begin{aligned}
x &= \frac{1}{2}+\frac{3}{8}+\sum_{n=4}^{\infty}\frac{n}{2^n}F_{n-2}+\sum_{n=4}^{\infty}\frac{n}{2^n}F_{n-3}\\
&= \frac{7}{8}+\frac{1}{2}\sum_{n=3}^{\infty}\frac{n+1}{2^n}F_{n-1}+\frac{1}{4}\sum_{n=3}^{\infty}\frac{n+2}{2^n}F_{n-1}\\
&= \frac{7}{8}+\left(\frac{1}{2}(x-\frac{2}{4}F_1)+\frac{1}{4}(S-\frac{1}{2}F_1)\right)+\left(\frac{1}{4}x+\frac{1}{4}S\right).
\end{aligned}$$

Now, applying (1) we have $S=2$ and therefore

$$x=\frac{3}{4}x+\frac{3}{2}$$

which implies $x=6$.

2. Solution 1

Denote $\angle CAB=\alpha$, $\angle ABC=\beta$, $\angle BCA=\gamma$. These correspond to arcs of the circumcircle, which correspond to the central angles $\angle BOC=2\alpha$, $\angle AOC=2\beta$, $\angle BOA=2\gamma$.

Denoting the area of any triangle $\triangle XYZ$ by S_{XYZ}, we conclude that the areas of the triangles $\triangle BOC$, $\triangle COA$, $\triangle AOB$ are, respectively,

$$S_{BOC}=\frac{1}{2}R^2\sin 2\alpha,\quad S_{COA}=\frac{1}{2}R^2\sin 2\beta,\quad S_{AOB}=\frac{1}{2}R^2\sin 2\gamma.$$

On the other hand, the area of each of these triangles equals to the radius of the incircle times half the perimeter. Therefore,

$$\begin{aligned}
S_{BOC} &= \frac{1}{2}R^2\sin 2\alpha\\
&= \frac{r_1}{2}(2R+BC)\\
&= \frac{r_1}{2}(2R+2R\sin\alpha)\\
&= r_1R(1+\sin\alpha).
\end{aligned}$$

which implies that

$$\frac{1}{r_1} = 2\frac{1+\sin\alpha}{R\sin 2\alpha} = \frac{1+\sin\alpha}{R\sin\alpha\cos\alpha} = \frac{1}{R}\left(\frac{1}{\cos\alpha} + \frac{2}{\sin 2\alpha}\right).$$

Similarly, we obtain

$$\frac{1}{r_2} = \frac{1}{R}\left(\frac{1}{\cos\beta} + \frac{2}{\sin 2\beta}\right), \quad \frac{1}{r_3} = \frac{1}{R}\left(\frac{1}{\cos\gamma} + \frac{2}{\sin 2\gamma}\right).$$

By summing up the last three equalities we get

$$\begin{aligned} & \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\ = \ & \frac{1}{R}\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} + \frac{1}{\cos\gamma}\right. \\ & \left. + \frac{2}{\sin 2\alpha} + \frac{2}{\sin 2\beta} + \frac{2}{\sin 2\gamma}\right). \end{aligned} \tag{1}$$

We now recall the following well known inequalities that hold for any triangle (for a proof, see below):

$$\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2} \tag{2}$$

$$\text{and} \quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \frac{3\sqrt{3}}{2}. \tag{3}$$

Applying the Cauchy-Schwartz inequality to the two triples

$$(\sqrt{\cos\alpha}, \sqrt{\cos\beta}, \sqrt{\cos\gamma}), \quad \left(\frac{1}{\sqrt{\cos\alpha}}, \frac{1}{\sqrt{\cos\beta}}, \frac{1}{\sqrt{\cos\gamma}}\right)$$

and using (2) gives

$$9 \leq (\cos\alpha + \cos\beta + \cos\gamma)\left(\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} + \frac{1}{\cos\gamma}\right),$$

which implies

$$\frac{1}{\cos\alpha} + \frac{1}{\cos\beta} + \frac{1}{\cos\gamma} \geq \frac{9}{\cos\alpha + \cos\beta + \cos\gamma} \geq \frac{9}{3/2} = 6. \tag{4}$$

By applying the Cauchy-Schwartz inequality to the triples

$$(\sqrt{\sin 2\alpha}, \sqrt{\sin 2\beta}, \sqrt{\sin 2\gamma}), \quad \left(\frac{1}{\sqrt{\sin 2\alpha}}, \frac{1}{\sqrt{\sin 2\beta}}, \frac{1}{\sqrt{\sin 2\gamma}}\right)$$

and using (3), it follows that

$$9 \leq (\sin 2\alpha + \sin 2\beta + \sin 2\gamma)\left(\frac{1}{\sin 2\alpha} + \frac{1}{\sin 2\beta} + \frac{1}{\sin 2\gamma}\right),$$

i.e.

$$\begin{aligned} \frac{1}{\sin 2\alpha} + \frac{1}{\sin 2\beta} + \frac{1}{\sin 2\gamma} &\geq \frac{9}{\sin 2\alpha + \sin 2\beta + \sin 2\gamma} \\ &\geq \frac{9}{3\sqrt{3}/2} \\ &= 2\sqrt{3}. \end{aligned} \tag{5}$$

Finally, using (4) and (5), with (1) we get

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \geq \frac{1}{R}(4\sqrt{3} + 6)$$

which is the required inequality.

It now remains to prove the two standard inequalities (2) and (3). The inequality (2) is obtained by

$$\begin{aligned} &\cos\alpha + \cos\beta + \cos\gamma = \cos\alpha + \cos\beta - \cos(\alpha + \beta) \\ = \;& 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \left(\cos^2\frac{\alpha+\beta}{2} - \sin^2\frac{\alpha+\beta}{2}\right) \\ = \;& 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 1 - 2\cos^2\frac{\alpha+\beta}{2} \\ \leq \;& 2\cos\frac{\alpha+\beta}{2} + 1 - 2\cos^2\frac{\alpha+\beta}{2}. \end{aligned}$$

The maximum of the quadratic polynomial $2x+1-2x^2$ is obtained for $x=\frac{1}{2}$, and its value is $\frac{3}{2}$. Thus, substituting $x=\cos\frac{\alpha+\beta}{2}$ we derive (2).

A straightforward method for proving (2) is to apply Jensen's inequality to the convex function $-\cos(x)$, and the result follows immediately.

To prove (3) we note that

$$\begin{aligned}\sin 2\alpha+\sin 2\beta+\sin 2\gamma &= 2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\ &\le 2\sin(\alpha+\beta)+2\sin\gamma\cos\gamma\\ &= 2(\sin\gamma+\sin\gamma\cos\gamma).\end{aligned}$$

Seeking the local maximum of the function

$$T(\gamma)=2(\sin\gamma+\sin\gamma\cos\gamma)$$

in $0\le\gamma\le\frac{\pi}{2}$ by differentiation, we get

$$\begin{aligned}2(\cos\gamma+\cos^2\gamma-\sin^2\gamma) &= 2(\cos\gamma+\cos 2\gamma)\\ &= 4\cos\frac{\gamma}{2}\cos\frac{3\gamma}{2}\\ &= 0\end{aligned}$$

(the derivative must vanish at the local maximum, since it is easy to verify that the maximum is not attained at the endpoints of the interval). The only solution of the latter equation in $\left[0,\frac{\pi}{2}\right]$ is $\gamma=60°$. Therefore, the required maximum is

$$2(\sin 60°+\sin 60°\cos 60°)=\frac{3\sqrt{3}}{2},$$

which completes the proof.

Solution 2

Let O be the circumcentre of $\triangle ABC$. Denote the distances of O from the sides $a=BC$, $b=CA$, $c=AB$ (of $\triangle ABC$) by d_a, d_b, d_c,

respectively. Denote the areas of $\triangle OBC$, $\triangle OCA$ and OAB by t_1, t_2, t_3, respectively and denote the area of $\triangle ABC$ by t.

Clearly, we have

$$\frac{1}{r_1} = \frac{2R+a}{2t_1} = \frac{R}{t_1} + \frac{a}{2t_1} = \frac{R}{t_1} + \frac{a}{ad_A} = \frac{R}{t_1} + \frac{1}{d_A},$$

and similarly

$$\frac{1}{r_2} = \frac{R}{t_2} + \frac{1}{d_B}; \quad \frac{1}{r_3} = \frac{R}{t_3} + \frac{1}{d_C}.,$$

Therefore,

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = R\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right) + \left(\frac{1}{d_A} + \frac{1}{d_B} + \frac{1}{d_C}\right).$$

Using the Cauchy-Schwartz inequality we have

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9$$

and obtain

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} \geq \frac{9}{t_1+t_2+t_3}$$

$$\text{and} \quad \frac{1}{d_A} + \frac{1}{d_B} + \frac{1}{d_C} \geq \frac{9}{d_A+d_B+d_C}.$$

Now by using the inequality $R \geq 2r$ and the equality

$$d_A + d_B + d_C = R + r \tag{6}$$

(see explanation below), it follows that

$$\frac{1}{d_A} + \frac{1}{d_B} + \frac{1}{d_C} \geq \frac{9}{R+r} \geq \frac{6}{R}.$$

Finally, we conclude that

Since

$$t \le \frac{3\sqrt{3}}{4}R^2$$

we obtain

$$\frac{1}{t} \ge \frac{4}{3\sqrt{3}R^2},$$

and therefore

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \ge \frac{36R}{3\sqrt{3}R^2} + \frac{6}{R} = \frac{4\sqrt{3}+6}{R}.$$

It now remains to prove (6). To that end, we recall that if r is the inradius of $\triangle ABC$ and S is its area, then $S = rp$, where

$$p = \frac{a+b+c}{2}$$

is half the perimeter of $\triangle ABC$.

Thus, we have

$$\begin{aligned}
p &= R(\sin\alpha + \sin\beta + \sin\gamma) \\
&= R(\sin\alpha + \sin\beta + \sin(\alpha+\beta)) \\
&= 2R\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}\right) \\
&= 4R\sin\frac{180^\circ-\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}
\end{aligned}$$

and

$$\begin{aligned}
d_A + d_B + d_C - R &= R(\cos\alpha + \cos\beta + \cos\gamma - 1) \\
&= R\left[\cos\alpha + \cos\beta - (\cos(\alpha+\beta) + 1)\right] \\
&= 2R\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right) \\
&= 4R\cos\frac{180^\circ-\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2} \\
&= 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (d_A + d_B + d_C - R)p \\
= \; & 16R^2 \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\gamma}{2} \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \cos\frac{\gamma}{2} \\
= \; & 16R^2 \sin\frac{\alpha}{2} \cos\frac{\alpha}{2} \cdot \sin\frac{\beta}{2} \cos\frac{\beta}{2} \cdot \sin\frac{\gamma}{2} \cos\frac{\gamma}{2} \\
= \; & 2R^2 \sin\alpha \sin\beta \sin\gamma \\
= \; & S.
\end{aligned}$$

and finally,

$$d_A + d_B + d_B - R = \frac{S}{p} = \frac{rp}{p} = r,$$

which implies (6).

3. We prove the statement by induction on m.

For $m = 1$ the statement is obvious.

Now choose n consecutive natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that each of the numbers $f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n)$ has at least m distinct prime factors.

Denote

$$A = [f(\alpha_1)f(\alpha_2)\ldots f(\alpha_n)]^2$$

and define

$$\beta_j = A + \alpha_j, \quad 1 \le j \le n.$$

Note that the numbers β_j are also n consecutive natural numbers. Now,

$$\begin{aligned}
f(\beta_j) &= a(A + \alpha_j)^2 + b(A + \alpha_j) + c \\
&= a\alpha_j^2 + b\alpha_j + c + A(aA + 2a\alpha_j + b) \\
&= f(\alpha_j) + (f(\alpha_1))^2 (f(\alpha_2))^2 \ldots (f(\alpha_n))^2 (aA + 2a\alpha_j + b) \\
&= f(\alpha_j) \left[1 + (f(\alpha_1))^2 (f(\alpha_2))^2\right]
\end{aligned}$$

$$\begin{aligned} & \ldots (f(\alpha_{j-1}))^2 f(\alpha_j)(f(\alpha_{j+1}))^2 \\ & \ldots (f(\alpha_n))^2 (aA + 2a\alpha_j + b)] \\ = \;& f(\alpha_j)[1 + Bf(\alpha_j)] \end{aligned}$$

for some integer B. Since $1+Bf(\alpha_j)$ and $f(\alpha_j)$ are relatively prime, it follows that $f(\beta_j)$ has at least one more prime factor than $f(\alpha_j)$ has. This completes the proof.

4. **Solution 1**

Assume that x and y are positive integer solutions of the equation

$$5^x - 3^y = 16.$$

It follows that

$$5^x \equiv 16 \pmod 3.$$

Consequently, $2^x \equiv 1 \pmod 3$. This is possible only if x is even, i.e., $x = 2a$ for some integer a.

Now,

$$5^x - 3^y = 16 \Rightarrow 1 - (-1)^y \equiv 0 \pmod 4$$

and this is possible only if y is even, i.e., $y = 2b$ for some integer b.

We can now conclude that

$$5^x - 3^y = 16 \Rightarrow 5^{2a} - 3^{2b} = 16 \Rightarrow (5^a)^2 - (3^b)^2 = 4^2. \quad (1)$$

Using (1), it follows that

$$(5^a + 3^b)(5^a - 3^b) = 16.$$

The only possibilities for factoring 16 are:

$$\mathrm{I}\begin{cases} 5^a + 3^b = 16 \\ 5^a - 3^b = 1 \end{cases} \qquad \mathrm{II}\begin{cases} 5^a + 3^b = 8 \\ 5^a - 3^b = 2 \end{cases}$$

From I it follows that $2 \cdot 5^a = 17$, which is impossible.

Option II yields $2 \cdot 5^a = 10$, i.e., $a = 1$ and therefore $b = 1$.

The only solution of the given equation is thus $x = y = 2$.

Solution 2

From (1) it follows that $5^a, 3^b, 4$ is a Pythagorean triple. It therefore has the representation

$$5^a = m^2 + n^2, \quad 3^b = m^2 - n^2, \quad 4 = 2mn,$$

for some integers m, n. From $3^b = m^2 - n^2$ it follows that $m^2 \geq n^2$ i.e., $m \geq n$. The equality $2mn = 4$ can hold only for $m = 2$, $n = 1$. Therefore,

$$\begin{aligned} &5^a = m^2 + n^2 = 5, \quad 3^b = m^2 - n^2 = 3 \Rightarrow \\ &a = b = 1 \Rightarrow x = 2a = 2, \quad y = 2b = 2, \end{aligned}$$

and the only solution is $x = y = 2$.

5. We denote

$$\angle AOB = \alpha, \quad \angle BOC = \beta, \quad \angle COD = \gamma.$$

Since $BC \parallel AO \parallel OD$ (where O is the centre of symmetry of the hexagon $ACBDEF$, and $AO = OD$, $BO = OE$, $CO = OF$), it follows that

$$\angle OBC = \alpha, \quad \angle BCO = \gamma$$

and that the quadrilateral $AOBC$ is a parallelogram. Therefore,

$$\angle OAB = \angle BCO = \gamma$$

which implies the congruence of the triangles OAB and OBC. Similarly, the triangles OBC and OCD are congruent. The other three triangles that have a common vertex O, namely $\triangle ODE$, $\triangle OEF$, $\triangle OFA$, have sides that are parallel to those of the triangles $\triangle OAB$, $\triangle OBC$ and $\triangle OCD$. By symmetry, they are of the same corresponding sides. It follows that all the triangles whose vertices are O plus two other consecutive vertices of the hexagon are congruent. By the construction of the triangles $\triangle FAF_1$, $\triangle EFE_1$, $\triangle DED_1$, $\triangle CDC_1$, $\triangle BCB_1$, $\triangle ABA_1$, we have

$$\angle OBB_1 = \alpha + 60^\circ, \quad \angle OBA_1 = \beta + 60^\circ,$$

and

$$\begin{aligned}\angle A_1BB_1 &= 360^\circ - \angle OBA_1 - \angle OBB_1\\ &= 180^\circ + \alpha + \beta + \gamma - (\beta + 60^\circ) - (\alpha + 60^\circ)\\ &= \gamma + 60^\circ.\end{aligned}$$

In $\triangle A_1BB_1$ and $\triangle OAA_1$ we have

$$OA = BC = BB_1, \quad AA_1 = A_1B,$$

and

$$\angle A_1AO = \angle A_1BB_1 = \gamma + 60^\circ.$$

Therefore, these two triangles are congruent, and $OA_1 = A_1B_1$. Similarly, the triangles $\triangle A_1BB_1$ and $\triangle OCB_1$ are also congruent and thus $OB_1 = A_1B_1$. This implies that the triangle $\triangle A_1OB_1$ is equilateral. If we pass over all the triangles having the common vertex O plus two other consecutive vertices of the hexagon $A_1B_1C_1D_1E_1F_1$, we conclude that they are all equilateral. Therefore, the hexagon $A_1B_1C_1D_1E_1F_1$ is regular.

We have so far proved that if $ABCDEF$ is an affinely regular hexagon, then $A_1B_1C_1D_1E_1F_1$ is a regular hexagon.

We now prove the opposite direction, namely, that if

$$A_1B_1C_1D_1E_1F_1$$

is a regular hexagon with centre O, then the triangles

$$\triangle OA_1B_1, \triangle OB_1C_1, \ldots, \triangle OF_1A_1$$

are equilateral and congruent. To that end, note that

$$\angle A_1B_1O = \angle B_1C_1O = \ldots = \angle F_1A_1O = 60^\circ.$$

Also,

$$\angle BB_1C = 60^\circ = \angle BB_1O + \angle OB_1C = 60^\circ - \angle A_1B_1B + \angle OB_1C.$$

Therefore, $\angle OB_1C = A_1B_1B$. Since $OB_1 = A_1B_1$, $B_1C = BB_1$, and it follows that $\triangle OB_1C$ and $\triangle A_1B_1B$ are congruent, which implies that $OC = A_1B = AB$. Similarly, we obtain that $OA = BC$, and thus $OABC$ is a parallelogram. Consequently,

$$OC = AB = OF, \qquad OC \parallel AB \parallel OF \parallel DE.$$

The analogous conclusion applies to the other sides and diagonals of $ABCDEF$, and this proves that $ABCDEF$ is affinely regular.

6. Solution 1

Let S be the set of all arrangements of the numbers $1, 2, \ldots, n$ in a row. The number of elements of S is $n!$.

We now define the function $f\colon S \to \Pi$ in the following way: for any given arrangement s there is a unique partition into blocks such that in each block the leftmost number is the largest, and those leftmost elements form a descending sequence. Finally, we define $f(s)$ as the partition of n whose summands are the number of elements of the blocks in this partition.

To illustrate, we give an example. Suppose that $n = 6$ and s is the arrangement $3, 1, 5, 2, 4, 6$. The blocks will be $\{3, 1\}$, $\{5, 2, 4\}$, $\{6\}$. Therefore, in this case $f(s)$ is the partition $6 = 2 + 3 + 1$.

Now, for any given partition $\alpha \in \Pi$ we compute the number of elements in $f^{-1}(\alpha)$. In other words, we compute the number of arrangements $s \in S$ for which $f(s) = \alpha$. The number of ways for distributing the n positions in the arrangement into sets of the sizes that occur in $\alpha = a_1(\alpha), a_2(\alpha), \ldots, a_n(\alpha)$, is

$$P(\alpha) = \frac{n!}{(1!)^{a_1(\alpha)}(2!)^{a_2(\alpha)} \cdots (n!)^{a_n(\alpha)} a_1(\alpha)! a_2(\alpha)! \cdots a_n(\alpha)!}.$$

After determining the positions distribution, the leftmost number in each block is uniquely determined. For the other positions in a block of size i we can arbitrarily arrange the other $i - 1$ numbers. The order of the blocks is also determined by the order of the leftmost elements. Consequently, the number of arrangements in $f^{-1}(\alpha)$ is

$$\begin{aligned} & P(\alpha) \cdot (0!)^{a_1(\alpha)}(1!)^{a_2(\alpha)} \cdot ((n-1)!)^{a_n(\alpha)} \\ = & \frac{n!}{1^{a_1(\alpha)} a_1(\alpha)! \cdot 2^{a_2(\alpha)} a_2(\alpha)! \cdots n^{a_n(\alpha)} a_n(\alpha)!}. \end{aligned}$$

Summing up this expression over all the partitions $\alpha \in \Pi$, counts actually the number of arrangements of the n elements, and we conclude that

$$\sum_{\alpha\in\Pi} \frac{n!}{1^{a_1(\alpha)}a_1(\alpha)!\cdot 2^{a_2(\alpha)}a_2(\alpha)!\cdots n^{a_n(\alpha)}a_n(\alpha)!} = n!.$$

Dividing by $n!$ gives the required result.

Solution 2

This solution uses generating functions.

We expand the infinite product

$$\prod_{n=1}^{\infty} e^{\frac{x^n}{n}} \tag{1}$$

into a power series in x. This yields

$$\prod_{n=1}^{N} e^{\frac{x^n}{n}} = \exp\left(\sum_{n=1}^{N}\frac{x^n}{n}\right). \tag{2}$$

For $0 < x < 1$, the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ is bounded by the geometric series $\sum_{n=1}^{\infty}x^n$, and it is therefore convergent. The series is obviously also convergent for $-1 < x < 0$. The explicit sum of this series is obtained by interchanging the order of summation and term by term integration of $\sum_{n=1}^{\infty}x^n$:

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\int_0^x t^{n-1}\,dt = \int_0^x \sum_{n=1}^{\infty} t^{n-1}\,dt = \int_0^x \frac{dt}{1-t} = -\ln(1-x).$$

From (2) we have

$$f(x) = \prod_{n=1}^{\infty} e^{\frac{x^n}{n}} = \lim_{N\to\infty}\prod_{n=1}^{N} e^{\frac{x^n}{n}} = \lim_{N\to\infty}\exp\left(\sum_{n=1}^{N}\frac{x^n}{n}\right).$$

From the continuity of the function e^t we have

$$\begin{aligned} f(x) &= \lim_{N\to\infty} \exp\left(\sum_{n=1}^{N} \frac{x^n}{n}\right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) \\ &= e^{-\ln(1-x)} \\ &= \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty} x^n. \end{aligned} \tag{3}$$

On the other hand, by expanding every term in the product (1) in a power series we obtain

$$e^{\frac{x^n}{n}} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^n}{n}\right)^k,$$

and therefore

$$f(x) = \sum_{a_1=0}^{\infty} \frac{x^{a_1}}{a_1!} \sum_{a_2=0}^{\infty} \frac{1}{a_2!}\frac{x^{2a_2}}{2^{a_2}} \cdots \sum_{a_n=0}^{\infty} \frac{1}{a_n!}\frac{x^{na_n}}{n^{a_n}} \cdots. \tag{4}$$

By comparing the coefficients of x^n in (4) with the corresponding coefficients in (3) we end up with

$$\begin{aligned} &\sum_{\Pi} \frac{1}{1^{a_1}a_1! \cdot 2^{a_2}a_2! \cdots n^{a_n}a_n!} = 1 \text{ where} \\ &\Pi = \{a_1, a_2, \ldots, a_n \mid a_1 + 2a_2 + \cdots + na_n = n\} \end{aligned} \tag{5}$$

To justify (5), it remains to show that the series obtained from the expansion (4) is convergent in the domain $|x| < 1$. To show that, we denote

$$f_N(x) = \prod_{n=1}^{N} e^{\frac{x^n}{n}} = \sum_{a_1=0}^{\infty} \frac{x^{a_1}}{a_1!} \sum_{a_2=0}^{\infty} \frac{1}{a_2!}\frac{x^{2a_2}}{2^{a_2}} \cdots \sum_{a_N=0}^{\infty} \frac{1}{a_N!}\frac{x^{Na_N}}{N^{a_N}}.$$

In this product the sum of the coefficients up to $n = N$ is

$$c_n = \sum_{\Pi} \frac{1}{1^{a_1} a_1! \cdot 2^{a_2} a_2! \cdots n^{a_n} a_n!}.$$

The coefficients c_n' of x^n in the expansion of $f_N(x)$ are all positive and satisfy

$$\begin{aligned} c_n' &= c_n, \quad n = 0, 1, \ldots, N, \\ c_n' &< c_n, \quad n > N. \end{aligned} \tag{6}$$

Also, $f_N(x)$ can be written as

$$f_N(x) = g_N(x) f(x), \quad g_N(x) = \exp\left(-\sum_{m=N+1}^{\infty} \frac{x^m}{m}\right).$$

Consequently, for $n \leq N$, the coefficients c_n' of x^n are obtained by applying Leibnitz's differentiation rule:

$$\begin{aligned} c_n' &= \frac{1}{n!} f_N^{(n)}(0) \\ &= \frac{1}{n!} [g_N f]^{(n)}(0) \\ &= \frac{1}{n!} \left[g_N f^{(n)}(0) + \sum_{k=1}^{n} g_N^{(k)}(0) f^{(n-k)}(0) \right]. \end{aligned}$$

All the derivatives of $g_N(x)$ up to order N vanish at $x = 0$. Therefore for $n \leq N$,

$$c_n' = \frac{1}{n!} f_N^{(n)}(0) = \frac{1}{n!} f^{(n)}(0) = \frac{1}{n!} \left(\frac{1}{1-x}\right)^{(n)}_{x=0} = 1.$$

From (6), it follows that we can change the order of summation in the series expansion of $f_N(x)$, and therefore (5) is justified.

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1. Let $l \geq 2$ be the degree of $f(x)$ and k_i — the degree of $g_i(x)$.

Statement: $k_i = l^i$ for $i = 1, 2, \ldots$

Proof: The proof is by induction. For $i = 1$ the statement is obvious.

Assume that $k_i = l^i$ and write

$$\begin{aligned} f(x) &= a_1 x^l + a_2 x^{l-1} + \cdots + a_{l+1} \\ \text{and} \quad g_i(x) &= b_1 x^{k_i} + b_2 x^{k_i - 1} + \cdots + b_{k_i+1}. \end{aligned}$$

Now

$$\begin{aligned} g_{i+1}(x) &= a_1 (b_1 x^{k_i} + b_2 x^{k_i-1} + \cdots + b_{k_i+1})^l \\ &\quad + a_2 (b_1 x^{k_i} + b_2 x^{k_i-1} + \cdots + b_{k_i+1})^{l-1} \\ &\quad + \cdots + a_{l+1}. \end{aligned} \tag{1}$$

Clear the leading term is of power $k_i l$, i.e. l^{i+1} as required.

In the polynomial $g_i(x)$ denote by a_{1i} the coefficient of the highest power of x and by a_{2i} the coefficient of the next lower power. Then by Viète formula the sum of the roots of $g_i(x)$ is $-a_{2i}/a_{1i}$, therefore

$$r_i = \frac{-a_{2i}/a_{1i}}{l^i}.$$

Given a_{1i}, a_{2i}, we show that the next two equalities hold:

$$a_{1,i+1} = s a_{1i}^l, \quad a_{2,i+1} = s l a_{1i}^{l-1} a_{2i},$$

where s is the coefficient of x^l in $f(x)$.

Proof: If

$$f(x) = s x^l + a_2 x^{l-1} + \cdots + a_{l+1}$$

then by (1) with $a_{1s} = s$ it follows that the coefficient of the highest power is obtained only from the first term, namely

$$a_{1,i+1} = s a_{1i}^l.$$

Since $l \geq 2$, the highest power of all the terms in the sum (1) is

$$l^i(l-1) = k_i(l-1) < k_i l - 1 = l^{i+1} - 1,$$

therefore the coefficient of $x^{l^{i+1}-1}$ in the polynomial $g_{i+1}(x)$ is the coefficient of $x^{l^{i+1}-1}$ in the first term of the sum (1), namely

$$a_{2,i+1} = s l a_{1i}^{l-1} a_{2i},$$

which proves the statement.

Now, given $r_i = \dfrac{-a_{2i}/a_{1i}}{l^i}$, it follows that

$$r_{i+1} = \frac{-a_{2,i+1}/a_{1,i+1}}{l^{i+1}} = -\frac{sla_{1i}^{l-1}a_{2i}/sa_{1i}^{l}}{l^{i+1}} = \frac{-a_{2i}/a_{1i}}{l^i} = r_i,$$

therefore

$$r_{99} = r_{98} = \ldots = r_{19} = 99.$$

2. The problem deals with angles between lines, and the discussed properties are invariant under translations of the lines. Therefore, with no loss of generality, we assume hereafter that all the lines pass through the origin.

Consider a pair of lines l_1, l_2 forming between them angles α and $180° - \alpha$. Assume that $r \leq n-1$ of the given $2n-1$ lines are in the domain of the angle α and $2n-r-1$ of them are in the domain of the angle $180° - \alpha$. One of the angles α and $180° - \alpha$ is obtuse. Therefore the number of triangles whose obtuse angle is between l_1 and l_2 is either r or $2n-r-1$, and this number is *at least* r. Counting all the possibilities of pairs of lines such that there are r other line "between" them, we see that there are $2n+1$ possibilities. Therefore the number of obtuse angled triangles is at least

$$(2n+1)\sum_{r=1}^{n-1} r = \frac{n(n-1)(2n+1)}{2}.$$

Hence the number of acute angled triangles is at most

$$\binom{2n+1}{3} - \frac{n(n-1)(2n+1)}{2} = \frac{n(n+1)(2n+1)}{6}.$$

Finally, to show that this bound is strict, it is to be proved that

$$N = \frac{n(n+1)(2n+1)}{6}$$

acute angled triangles can be obtained.

It is easy to see that if the angles between them are all equal (and equal to $\frac{360°}{2n+1}$), then N triangles are actually obtained.

3. Put $x = y = 0$ and obtain

$$f(0) = f(0)^2 - f(0) + 1 \Rightarrow (f(0)-1)^2 = 0 \Rightarrow f(0) = 1.$$

Now put $y = -x$ and obtain,

$$1 = f(x)f(-x) - f(-x^2) + 1 \Rightarrow f(x)f(-x) = f(-x^2),$$

and in the last equation, putting $x = 1$, we obtain

$$f(-1)(f(1) - 1) = 0,$$

i.e. $f(1) = 1$ or $f(-1) = 0$.

Now, if $f(1) = 1$, putting in the last equation $y = 1$ we obtain

$$f(x+1) = f(1)f(x) - f(x) + 1 = f(x) - f(x) + 1,$$

i.e. $f(x) = 1$ for every x. Such a function satisfies in fact the requirement of the problem, since $1 = 1 - 1 + 1$.

Assume now $f(-1) = 0$. Putting $y = -1$ in the first equation, we obtain

$$f(x-1) = -f(-x) + 1,$$

i.e.

$$f(x-1) + f(-x) = -1 \quad \forall x, \tag{1}$$

In addition, putting $y = 1$, we obtain

$$f(x+1) = (f(1) - 1)f(x) + 1. \tag{2}$$

Put $x = -1$ in (1) and $x = -2$ in (2) and obtain

$$f(1) + f(-2) = 1, \quad 0 = f(-2)(f(1) - 1) + 1.$$

The second equation yields $f(-2)f(1) + 1 - f(-2) = 0$ and since $1 - f(-2) = f(1)$, it follows that $f(1)(2 - f(1)) = 0$, i.e. $f(1) = 0$ or $f(1) = 2$.

Assume $f(1) = 0$. Then by (1), $f(x+1) = -f(x)+1$, and replacing $x + 1$ by x, we obtain $f(x+2) = -f(x+1) + 1 = f(x)$, i.e. $f(x+2) = f(x)$ for every x. But putting $y = 2$ in the first equation it follows that

$$f(x+2) = f(2)f(x) - f(2x) + 1,$$

and by $f(2) = f(0) = 1$ it follows

$$f(x) = f(x) - f(2x) + 1 \Rightarrow f(2x) = 1.$$

Hence $f(x) = 1$ for every x, contrary to $f(1) = 0$, and therefore $f(1) = 2$. In such a case (1) becomes $f(x+1) = f(x) + 1$ and by induction we find that $f(n) = n + 1$ and

$$f(x+n) = f(x) + n \tag{3}$$

for each integer n and any x.

Now we prove that

$$f\left(\frac{m}{n}\right) = \frac{m}{n} + 1$$

for any integers m, n and this will imply that $f(x) = x+1$ for every rational x.

Putting $x = \dfrac{m}{n}$, $y = n$ in the first equation yields

$$f\left(\frac{m}{n} + n\right) = f(n)f\left(\frac{m}{n}\right) - f(m) + 1.$$

By (3)

$$f\left(\frac{m}{n} + n\right) = n + f\left(\frac{m}{n}\right),$$

and since $f(n) = n + 1$,

$$\begin{aligned} n + f\left(\frac{m}{n}\right) &= nf\left(\frac{m}{n}\right) + f\left(\frac{m}{n}\right) - (1+m) + 1 \\ \Rightarrow \quad nf\left(\frac{m}{n}\right) &= n + m \\ \Rightarrow \quad f\left(\frac{m}{n}\right) &= \frac{n+m}{n} = 1 + \frac{m}{n}. \end{aligned}$$

Hence $f(x) = x + 1$. It remains to show that this function satisfies the conditions of the problem. In fact,

$$f(x+y) = x + y + 1,$$

leading to

$$\begin{aligned} f(x+y) &= f(x)f(y) - f(xy) + 1 \\ &= (x+1)(y+1) - (xy+1) + 1 \\ &= x + y + 1, \end{aligned}$$

therefore there aretwo solutions, $f(x) = 1$ and $f(x) = x + 1$.

4. We prove that $a_{n+1} = 2ca_n - a_{n-1}$ for every $n \geq 2$.

Proof: By definition,

$$a_n = ca_{n-1} + \sqrt{(c^2-1)(a_{n-1}^2 - 1)}.$$

Transferring from one side to another and squaring we obtain

$$\begin{aligned}(a_n - ca_{n-1})^2 &= a_n^2 - 2ca_na_{n-1} + c^2a_{n-1}^2 \\ &= c^2a_{n-1}^2 - c^2 - a_{n-1}^2 + 1\end{aligned}$$

which by cancelling by $c^2a_{n-1}^2$ and adding $c^2a_n^2$ at both sides yields

$$c^2a_n^2 - a_n^2 - c^2 + 1 = c^2a_n^2 - 2ca_na_{n-1} + a_{n-1}^2,$$

i.e.

$$(c^2 - 1)(a_n^2 - 1) = (ca_n - a_{n-1})^2.$$

Now $ca_n - a_{n-1} > 0$ since $c > 0$ and by definition $a_n \geq a_{n-1}$, therefore

$$\sqrt{(c^2 - 1)(a_n^2 - 1)} = ca_n - a_{n-1}$$

and adding ca_n to each side yields

$$a_{n+1} = ca_n + \sqrt{(c^2 - 1)(a_n^2 - 1)} = 2ca_n - a_{n-1}.$$

We shall prove by induction that $a_n \in N$ for each n.

First

$$a_1 = c \in N, \quad a_2 = c^2 + \sqrt{(c^2 - 1)(c^2 - 1)} = 2c^2 - 1 \in N.$$

Now if $a_n \in N$ and $a_{n+1} \in N$, then also

$$a_{n+2} = 2ca_{n+1} - a_n \in N.$$

5. Take a point $P = (t, t^2 - t, 0)$ for some t. It is easy to see that

$$f(t, t^2 - t, 0) = (t^4 + 2t^2 - 2t^2)/t^2 = 2 + t^2 - 2t.$$

We restrict our discussion to t such that $0 < t < 2$ so that

$$t^2 - 2t < 0.$$

It is necessary to solve the inequality $t^2 - 2t + 0.001 > 0$, and smaller values of t are in the interval

$$0 < t < 1 - 3/100\sqrt{1110}.$$

$t = 1/10000 = 0.0001$ is of course in this interval, since

$$\frac{3333}{100} > \sqrt{1110}.$$

Now for $t = 1/10000$ we obtain clearly

$$x_0^2 + y_0^2 + z_0^2 = (t^4 + 2t^2 - 2t^3) < \frac{1}{2000}.$$

Note that by the choice $P = (t, t^2 - t, 0)$ for some t and computation of $f(P)$ at such a point we conclude existence of an appropriate point, but our problem was to find such a point. Also note that 2 **is not** a limit of $f(x, y, z)$ at $(0, 0, 0)$ since f has no limit at this point.

6. We show first that the number of students cannot exceed 9. Suppose negatively that there are more than nine students in the set. Consider their answers to question 1. Note that there are only three possibilities. Hence, by the Pigeon Hole principle, we obtain that there is a set of at least seven students whose answers are from a subset of two choices only.

We choose now such a 7-tuple of students and consider their answers to question 2. There are among them a subset of (at least) five students whose answers are from a subset of two choices only. In this 5-tuple there is a subset of (at least) four students which chose from two choices only for question 3. Therefore it follows that for satisfying the conditions of the problem these four students have to give different answers to question 4. This is impossible, since there are only three choices.

Consequently the number of students does not exceed 9. To finish the solution, we show that there exists a set of nine students with three choices a, b, c for question 1, 2, 3, 4 such that the condition of the problem is satisfied. One possibility is:

Student	1	2	3	4	5	6	7	8	9
Question 1	a	a	a	b	b	b	c	c	c
Question 2	a	b	c	a	b	c	a	b	c
Question 3	a	b	c	b	c	a	c	a	b
Question 4	a	c	b	b	a	c	c	b	a

2000

1. Denote the number of summands in s by $N(s)$, and the maximal summand in s by $M(s)$. Then clearly

$$f(s) = N(s) + M(s) \geq 2\sqrt{N(s) \times M(s)} \geq 2(\sqrt{2000}) > 89. \quad (1)$$

Explanation: The first inequality is the G-E inequality. To obtain the second inequality, note that by replacing each summand in a partition of 2000 with $M(s)$ we obtain a larger sum (unless all the elements are equal). The new sum is $N(s) \times M(s)$.

Since $f(s) = N(s) + M(s)$ is an integer, it follows by (1) that $N(s) + M(s) \geq 90$.

The minimal value 90 is attained for the partition

$$s_0 = \underbrace{45 + 45 + \ldots + 45}_{44 \text{ times}} + 20$$

or for the partition

$$s_0 = \underbrace{40 + 40 + \ldots + 40}_{50 \text{ times}}.$$

Clearly, $N(s_0) = 45$, $M(s_0) = 45$ and therefore $f(s_0) = 90$. Consequently, the required answer is 90.

2. First solution:

The statement is false. To prove this, take $k = 4$ and assume by contradiction that there exists a positive integer n for which $\binom{n}{i}$ is divisible by 4 for every $1 \leq i \leq n - 1$. Then,

$$\sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2$$

is divisible by 4. This is false for every $n > 1$, and thus we arrive at a contradiction.

Second solution:

We prove that the set of positive integers k for which the claim holds is exactly the set of primes. Clearly, if k is a prime, then we can take $n = k$. For every $1 \leq i \leq k - 1$, the numerator of

$$\frac{k!}{i!(k-i)!}$$

is divisible by k, while its denominator is not divisible by k. Since k is prime, it follows that $\binom{k}{i}$ is divisible by k.

Suppose now that k is not a prime. Then consider two cases:

First case: $k = p^r$, where p is a prime and $r > 1$.

We find a value of i for which the statement does not hold.

Suppose that there is a positive integer n such that for all $1 \le i \le n-1$, $\binom{n}{i}$ is divisible by p^r. Obviously, n is divisible by p^r, and we write $n = p^\alpha \beta$ for some β with $gcd(\beta, p) = 1$.

Take $i = p^{\alpha-1}$. Then,

$$\binom{n}{i} = \prod_{j=0}^{p^{\alpha-1}-1} \frac{\beta p^\alpha - j}{p^{\alpha-1} - j}.$$

When $j = 0$ we have

$$\frac{\beta p^\alpha - j}{p^{\alpha-1} - j} = \beta p.$$

When j is coprime with p, both the above numerator and the denominator are coprime with p.

In all other cases, we write $j = \delta p^\gamma$ for some δ coprime with p and $\gamma \le \alpha - 2$. Thus,

$$\frac{\beta p^\alpha - j}{p^{\alpha-1} - j} = \frac{\beta p^\alpha - \delta p^\gamma}{p^{\alpha-1} - \delta p^\gamma} = \frac{p^\gamma(\beta p^{\alpha-\gamma} - \delta)}{p^\gamma(p^{\alpha-\gamma-1} - \delta)}.$$

Now, since $\alpha-\gamma-1 \ge 1$, we have $\beta p^{\alpha-\gamma} - \delta$ and $p^{\alpha-\gamma-1} - \delta$ coprime with p. In this case, the power of p in the above numerator and the denominator is γ, and the power of p in the above product of fractions, which is an integer, is 1. This contradicts the assumption that $p^r \mid n$.

Second case: k is divisible by at least two distinct primes p, q.

Assume by contradiction that there is a positive integer n as required. Then n is divisible by pq and we can write $n = p^\alpha \beta$ where $\gcd(p, \beta) = 1$ and $\beta > 1$ (since q divides β). Take $i = p^\alpha$. Then,

$$\binom{n}{i} = \prod_{j=0}^{p^\alpha-1} \frac{\beta p^\alpha - j}{p^\alpha - j}.$$

When $j = 0$,

$$\frac{\beta p^\alpha - j}{p^\alpha - j} = \beta$$

is co-prime with p. In all other cases, both the numerator and the denominator of

$$\frac{\beta p^\alpha - j}{p^\alpha - j}$$

are either co-prime with p or are divisible by the same power of p, and therefore the product of those fractions is not divisible by p. But p divides k, and hence $\binom{n}{i}$ is not divisible by k, contrary to our assumption.

Finally, the only positive integers k for which the claim holds are the primes.

3. **First solution:**

Let ABC be the triangle, I be its incentre, O be its circumcentre. The triangle itself is not equilateral to make the line IO well defined. Let J be the external centre similarity of the two circles and enlarge the incircle into the circumcircle from J.

Under this mapping $I \to O$, $A_1 \to A_2$, $B_1 \to B_2$ and $C_1 \to C_2$. Since OA_2 is parallel to IA_1, A_2 is the midpoint of the arc BC and similarly B_2 and C_2 are the midpoints of the corresponding arcs CA and AB. Hence AA_2, BB_2 and CC_2 are bisecting the angles A, B and C and thus these lines meet at I.

Now AA_2 and C_2B_2 are perpendicular — this follows from the fact that the sum of the arcs AB_2 and A_2C_2 is the semiperimeter of the circumcircle — so AA_2 is the altitude of the triangle $A_2B_2C_2$. Similarly, BB_2 and CC_2 are also altitudes so I is the orthocentre of the triangle $A_2B_2C_2$. Hence I is the image of the orthocentre M of triangle $A_1B_1C_1$ under the above enlargement. Thus JI is passing through M and O and we are done.

Second solution:

Denote by I, O the incentre and circumcentre of triangle ABC.

Inverse the figure with respect to the incircle.

Under this inversion, the midpoints of A_1B_1, B_1C_1, C_1A_1 are mapped to C, A, B, and thus the nine-points circle of triangle $A_1B_1C_1$ is mapped to the circumcircle of triangle ABC. Denote the centre of the nine-points circle of the triangle $A_1B_1C_1$ by T. Then I, O, T lie on a straight line (although O is not necessarily the image of T under the inversion). But the line passing through I, T is the Euler line of the triangle $A_1B_1C_1$, and therefore it passes through its orthocentre M. Hence I, M, O are collinear.

4. Look at the function $F\colon A \times B \to \{2, 3, \ldots, 4000\}$ defined by

$$F(a, b) = a + b.$$

We deal with two cases:

1. F is onto. Then both A and B contain 1 and 2000. Therefore, $1999 = 2000 - 1$ belongs to $(A - A) \cup (B - B)$.

2. F is not onto. Since $|A| \times |B| \geq 3999$, it follows by the pigeonhole principle that F cannot be one-to-one (by definition F may have at most 3999 possible values and F does not assume all of them). So, we must have a_1, a_2 in A and b_1, b_2 in B such that $a_1 + b_1 = a_2 + b_2$. From this we get $a_1 - a_2 = b_2 - b_1$, and since $a_1 - a_2 \in A - A$ and $b_2 - b_1 \in B - B$, it follows that $A - A \cap B - B$ is nonempty, which completes the proof.

5. It is easy to verify that the product of two numbers of the form $m^2 + dn^2$ can also be written as follows:

$$(x^2 + dy^2)(u^2 + dv^2) = (xu \pm dyv)^2 + d(xv \mp yu)^2, \qquad (1)$$

i.e. also in the form $m^2 + dn^2$.

Now let $q = (a^2 + db^2)$ and $p = x^2 + dy^2$ be the corresponding representations of q and p. Write $q = rp$. We are going to show that there exist integers u, v such that a and b can be written either as

$$\begin{cases} a = xu + dyv \\ b = xv - yu \end{cases} \qquad (2)$$

or

$$\begin{cases} a = xu - dyv \\ b = xv + yu \end{cases} \qquad (3)$$

Elimination of v from the system (2) yields

$$v = \frac{ay + bx}{x^2 + dy^2}$$

and from the system (3)

$$v = \frac{ay - bx}{x^2 + dy^2}.$$

Thus v is an integer if and only if either $x^2 + dy^2 | ay + bx$ or $x^2 + dy^2 | ay - bx$.

Since $x^2 + dy^2$ is prime, it is enough to show that

$$x^2 + dy^2 | (ay + bx)(ay - bx). \tag{4}$$

Let

$$T = (ay + bx)(ay - bx).$$

Then

$$T = (a^2 + db^2)y^2 - (x^2 + dy^2)b^2.$$

Since $x^2+dy^2 | a^2+db^2$, the product T is indeed divisible by x^2+dy^2. Thus in one of the systems (2), (3) v is a whole number. Each of these systems implies that u is rational. On the other hand

$$u^2 + dv^2 = \frac{a^2 + db^2}{x^2 + dy^y} = r$$

is an integer, and if v is an integer, it follows that u is also an integer and we have finished.

6. First solution:

Define

$$b_j = \sum_{i=1}^{k} a_{ij}^p$$

Denote the left hand side of the required inequality by L and its right hand side by R. Then

$$L^q = \sum_{j=1}^{l} b_j^{q/p} = \sum_{j=1}^{l} b_j^{(q-p)/p} \left(\sum_{i=1}^{k} a_{ij}^p\right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{l} b_j^{(q-p)/p} a_{ij}^p\right).$$

Using Hölder's inequality it follows that

$$\begin{aligned} L^q &\le \sum_{i=1}^{k} \left[\left(\sum_{j=1}^{l} (b_j^{(q-p)/p})^{q/(q-p)}\right)^{(q-p)/q} \left(\sum_{j=1}^{l} (a_{ij}^p)^{q/p}\right)^{p/q} \right] \\ &= \sum_{i=1}^{k} \left[\left(\sum_{j=1}^{l} b_j^{q/p}\right)^{(q-p)/q} \left(\sum_{j=1}^{l} a_{ij}^q\right)^{p/q} \right] \\ &= \left(\sum_{j=1}^{l} b_j^{q/p}\right)^{(q-p)/q} \cdot \left[\sum_{i=1}^{k} \left(\sum_{j=1}^{l} a_{ij}^q\right)^{p/q} \right] \\ &= L^{q-p} R^p. \end{aligned}$$

The inequality $L \leq R$ follows by dividing the last inequality by L^{q-p} and taking the p-th root.

Hölder's inequality states:

Let p, q positive numbers such that $p > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then, for every two sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ of nonnegative real numbers,

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}$$

Second solution:

Denote $r = \frac{q}{p}$, $b_{ij} = a_{ij}^p$. Then $r \geq 1$, and the given inequality is equivalent to the following inequality:

$$\left(\sum_{j=1}^{l}\left(\sum_{i=1}^{k} b_{ij}\right)^r\right)^{\frac{1}{r}} \leq \sum_{i=1}^{k}\left(\sum_{j=1}^{l} b_{ij}^r\right)^{\frac{1}{r}}$$

We shall prove this inequality by induction on k. For $k = 1$, we have equality. For $k = 2$, the inequality becomes Minkowsky's inequality.

Suppose that $k \geq 3$ the inequality holds for $k - 1$. Then by the induction assumption for $k - 1$ we have

$$\sum_{i=1}^{k}\left(\sum_{j=1}^{l} b_{ij}^r\right)^{\frac{1}{r}} \geq \left(\sum_{j=1}^{l}\left(\sum_{i=1}^{k-1} b_{ij}\right)^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{l} b_{kj}^r\right)^{\frac{1}{r}}$$

Using the induction assumption for $k = 2$ (or Minkowsky's inequality) with

$$\tilde{b_{1j}} = \sum_{i=1}^{k-1} b_{ij}, \qquad \tilde{b_{2j}} = b_{kj},$$

we have

$$\left(\sum_{j=1}^{l}\left(\sum_{i=1}^{k-1} b_{ij}\right)^r\right)^{\frac{1}{r}} + \left(\sum_{j=1}^{l} b_{kj}^r\right)^{\frac{1}{r}} \geq \left(\sum_{j=1}^{l}\left(\sum_{i=1}^{k} b_{ij}\right)^r\right)^{\frac{1}{r}}$$

And we are done.

Minkowsky's inequality states:

For any $a_i, b_i \geq 0$, $i = 1, 2, \ldots, n$ and $p > 1$,

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}$$

(for $p < 1$ reverse the direction of the inequality). Equality holds if and only if $b_k = \lambda a_k$ for some λ and for all k.

2001

1. **First Solution:**

We shall use the identity

$$n(m^2+2m-n)^2+(m^2-2mn+n)^2=(n+1)(m^2+n)^2.$$

Which can be easily proven. Setting $n=2000$, $m=90000$ will give

$$x=m^2+2m-n>m^2+n=z,$$

$$\begin{aligned} z &= m^2+n \\ &= (45\cdot 2000)^2+2000 \\ &> 2000^2\cdot 2025 \\ &> 1999\cdot 2000\cdot 2001, \end{aligned}$$

and

$$\begin{aligned} y &= m^2-2mn+n \\ &= (45\cdot 2000)^2-2\cdot 45\cdot 2000+2000 \\ &< 2000^3-2000 \\ &= 1999\cdot 2000\cdot 2001. \end{aligned}$$

And thus the triple x, y, z satisfies the condition of the problem.

One can ask how to come up with such an identity. One way is to note that the equation

$$nx^2+y^2=(n+1)^2$$

is equivalent to

$$n(x-y)(x+y)=(n+1)(z-y)(z+y).$$

We can set

$$x=y+2n(m+1), z=y+2m(n+1)$$

and the equation becomes

$$4nm(n+1)(y+mn+m)=4n(m+1)(n+1)(y+mn+n).$$

Extracting y we get

$$y=m^2-2mn+n,$$

and thus

$$x = m^2 + 2m - n, \qquad z = m^2 + n.$$

Second Solution:

Here we offer a more constructive method for a solution. We shall look for a solution with $y = 1$. Consider the Pell equation

$$X^2 - 2000 \cdot 2001 Z^2 = -2000$$

It has a basic solution $(2000, 1)$. The unit Pell equation

$$X^2 - 2000 \cdot 2001 Z^2 = 1$$

has a solution $(4001, 2)$. Therefore, by Pell's theorem, we have an infinite family of solutions, namely

$$\begin{aligned} & X_n + Z_n\sqrt{2000 \cdot 2001} \\ = \ & (2000 + \sqrt{2000 \cdot 2001})(4001 + 2\sqrt{2000 \cdot 2001})^n. \end{aligned}$$

Now it is simply a matter of computation of the first three solutions. We arrive at the following results:

$$\begin{aligned} (X_1, Z_1) &= (2000 \cdot 8003, 8001), \\ (X_2, Z_2) &= (2000[8003 \cdot 4001 + 4002 \cdot 8001], \\ & \qquad 4000 \cdot 8003 + 4001 \cdot 8001), \\ (X_3, Z_3) &= (2000[8003 \cdot 4001^2 + 2 \cdot 4001 \cdot 4002 \cdot 8001 \\ & \quad +4000 \cdot 4002 \cdot 8003], 4001 \cdot 4000 \cdot 8003 \\ & \quad +8001 \cdot 4001^2 + 4000 \cdot 4001 \cdot 8003 \\ & \quad +4000 \cdot 4002 \cdot 8001). \end{aligned}$$

It turns out that

$$(x, y, z) = (\frac{X_3}{2000}, 1, Z_3)$$

satisfies the conditions of the problem. It is clear that $x > z$ since

$$2000x^2 - 2001z^2 = -1.$$

Further,

$$y = 1 < 1999 \cdot 2000 \cdot 2001.$$

Finally,

$$z > 4001 \cdot 4000 \cdot 8003 > 1999 \cdot 2000 \cdot 2001.$$

And we are finished.

2. First Solution:

Let P be a point on the locus.

If $P \in l$, then P lies either inside BC, and then

$$\angle APB = \angle CPD = 180^\circ$$

or outside AD, and then

$$\angle APB = \angle CPD = 0^\circ.$$

Assume from now on that P does not lie on l. Let c be the circle such that the inversion with respect to it maps A, B to D, C. It is well known that there exists a single circle with this property. The centre of c lies on the line l. This centre could be at the point of infinity and in this case c is a line. The latter case occurs if and only if $AB = CD$.

Denote by c_1, c_2 the circumcircles of $\triangle PAB, \triangle PCD$ respectively. Let P' be the invariant point of P under the inversion with respect to c. Then $\triangle P'CD \sim \triangle PBA$ and therefore

$$\angle CPD = \angle APB = \angle CP'D$$

so P' lies on c_2. Hence the inversion with respect to c maps c_1 to c_2, and hence they intersect on c. We conclude that P lies on c.

Conversely, suppose that P lies on C. Then $\triangle PAB \sim \triangle PDC$ and therefore $\angle APB = \angle CPD$, so P lies on the locus. We conclude that the locus is the circle c, the segment BC and the outside of the segment AD.

Second Solution:

The case where P lies on l is treated the same as in the last solution, and we shall ignore this case from now on.

Let PM be the bisector of $\angle BPC$, with M on l. Then PM is the bisector of $\angle APD$ as well. Denote the area of the triangle $\triangle XYZ$ by $S_{\triangle XYZ}$. Then

$$\frac{AB}{CD} = \frac{S_{\triangle ABP}}{S_{\triangle CDP}} = \frac{PA \cdot PB \cdot \sin \angle APB}{PC \cdot PD \cdot \sin \angle CPD} = \frac{MA \cdot MB}{MC \cdot MD}.$$

The function

$$f(M) = \frac{MA \cdot MB}{MC \cdot MD}$$

is continuous and monotonic on the segment BC.

Since $f(B) = 0, f(C) = \infty$ then there exists a unique point M such that

$$f(M) = \frac{AB}{CD},$$

so the bisector of $\angle BPC$ meets BC at a unique point M. Then P lies on the circle of Apollonius of B, C with respect to this M, which could also be a line if $AB = CD$ (and then M is the midpoint of BC).

If P lies on the circle of Apollonius of the point M satisfying

$$f(M) = \frac{AB}{CD},$$

then by the formulas for the areas of the triangles $\triangle ABP$, $\triangle CDP$ we have $\angle APB = \angle CPD$.

Third Solution:

We shall use analytic geometry. Fix a coordinate system where l is the x-axis, the point $(0, 0)$ being the point O lying inside BC for which $OA \cdot OB = OC \cdot OD$ and fix the unit length to be this product. Let

$$A = (a, 0), \quad B = (\frac{1}{a}, 0), \quad C = (c, 0), \quad D = (\frac{1}{c}, 0).$$

Then $a < \frac{1}{a} < 0 < c < \frac{1}{c}$, which implies $a < -1 < 0 < c < 1$.

Let P be a point on the locus, and let m_a, m_b, m_c, m_d be the slopes of PA, PB, PC, PD respectively. Then P is on the locus if and only if

$$\frac{m_a - m_b}{1 + m_a m_b} = \frac{m_c - m_d}{1 + m_c m_d}.$$

Let $P = (x, y)$. The latter equation becomes

$$\frac{\dfrac{y}{x-a} - \dfrac{y}{x - \frac{1}{a}}}{1 + \dfrac{y^2}{(x-a)(x - \frac{1}{a})}} = \frac{\dfrac{y}{x-c} - \dfrac{y}{x - \frac{1}{c}}}{1 + \dfrac{y^2}{(x-c)(x - \frac{1}{c})}}.$$

Now we want to divide by y both sides of the equation. This can only be done in case $y \neq 0$, i.e. P does not lie on l. The case where P lies on l is treated as in the first solution. Dividing by y and rearranging both sides, we get

$$\frac{a - \frac{1}{a}}{(x-a)(x - \frac{1}{a}) + y^2} = \frac{c - \frac{1}{c}}{(x-c)(x - \frac{1}{c}) + y^2}.$$

Multiplying by the common denominator and rearranging, we have

$$\left(a-\frac{1}{a}-c+\frac{1}{c}\right)(x^2+y^2+1)-2\left(\frac{a}{c}-\frac{c}{a}\right)x=0.$$

Suppose $ac=-1$, so $\frac{1}{a}-a=\frac{1}{c}-c$, or $AB=CD$, then the equation becomes

$$(a^2-c^2)x=0$$

$a^2\neq c^2$ for otherwise $a^2=c^2=1$ so the locus is the line $x=0$.

Now suppose $ac\neq -1$. Then we can divide by $(a-\frac{1}{a}-c+\frac{1}{c})$:

$$\left(x-\frac{a+c}{ac+1}\right)^2+y^2=\frac{(a^2-1)(1-c^2)}{(ac+1)^2}.$$

This is an equation of a circle. The radius is clearly positive since $a<-1, 0<c<1$.

It remains to check that if P satisfies the equation we have found, then it lies on the locus. But this is simply a matter of reversing the steps, and then we are finished.

3. We begin with a few properties of f:

$f(x)=f(y)$ implies

$$f(x)+x=f(f(x))=f(f(y))=f(y)+y$$

and therefore $x=y$. Hence f is $1-1$. We deduce that f is monotonic.

Let $a=f(0)$. Then $f(a)=a$ and thus

$$f(a)=f(f(a))=f(a)+a$$

so $a=0$.

By induction, one can prove that

$$f^{(n)}(x)=F_{n-1}x+F_nf(x)$$

where F_n is the Fibonacci sequence defined by $F_0=0$, $F_1=1$, and

$$F_{n+2}=F_{n+1}+F_n.$$

Consider now two cases:

Case 1: f is decreasing. Since $f(0)=0$, $x<0$ implies $f(x)>0$ and hence

$$0>f(f(x))=f(x)+x.$$

$x > 0$ implies $f(x) < 0$ and

$$0 < f(f(x)) = f(x) + x.$$

Both cases lead to $|f(x)| < |x|$, so

$$\frac{|f^{(n)}(x)|}{|x|} < 1.$$

Then for $x \neq 0$,

$$\frac{f(x)}{x} + \frac{F_{n-1}}{F_n} = \frac{1}{F_n} \cdot \frac{f^{(n)}}{x} \rightarrow_{n\rightarrow\infty} 0.$$

And therefore

$$f(x) = \lim_{n\rightarrow\infty} -\frac{F_{n-1}}{F_n}x = \frac{1-\sqrt{5}}{2}x.$$

Case 2: f is increasing. Then the sign of $f(x)$ is the same as the sign x (since $f(0) = 0$) and therefore

$$|f(f(x))| = |f(x)| + |x| \geq |x|.$$

Hence the f is onto, and since we have already shown that f is $1-1$ then f is bijective. Let $g = f^{-1}$. Then g is continuous and satisfies

$$g(g(x)) + g(x) = x.$$

By induction,

$$(-1)^{n+1}g^{(n)}(x) = F_n g(x) - F_{n-1}x.$$

The sign of $g(x)$ is also the same as the sign of x, and hence

$$|x| = |g(x)| + |g(g(x))|$$

which leads to

$$|g(x)| < |x|.$$

Therefore

$$\frac{|g^{(n)}(x)|}{|x|} < 1,$$

and thus for $x \neq 0$,

$$\frac{g(x)}{x} - \frac{F_{n-1}}{F_n} = (-1)^{n+1} \cdot \frac{1}{F_n} \cdot \frac{g^{(n)}(x)}{x} \rightarrow_{n\rightarrow\infty} 0.$$

Therefore

$$g(x) = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} x = \frac{2}{1+\sqrt{5}}$$

and

$$f(x) = \frac{1+\sqrt{5}}{2} x.$$

4. **First Solution:**

Let a be a root of P. Then $-1 = a^3 - 3a$. Raised by the fifth power, this yields

$$\begin{aligned} -1 &= a^5(a^2-3)^5 \\ &= a^5(a^{10} - 3^5 - 15a^2(a^2-3)(a^4-3a^2+9)). \end{aligned}$$

Since $a^3 - 3a = -1$

$$\begin{aligned} -1 &= a^5(a^{10} - 3^5 + 15(a^5 - 3(a^3-3a)) \\ &= a^5(a^{10} - 3^5 + 15(a^5+3)). \end{aligned}$$

Let $b = a^5$. Then

$$b^3 + 15b^2 - 198b + 1 = 0$$

So $Q(x) = x^3 + 15x^2 - 198x + 1$.

Second Solution:

Let a, b and c the roots of P. By the Vièta formulas,

$$\begin{aligned} a+b+c &= 0, \\ ab+bc+ca &= -3, \\ \text{and} \qquad abc &= -1. \end{aligned}$$

We want to find the coefficients of Q, given by

$$Q(x) = x^3 - (a^5+b^5+c^5)x^2 + (a^5b^5+b^5c^5+c^5a^5)x - a^5b^5c^5.$$

The last coefficient obviously equals -1. For the other two, we shall use Newton's formulas for $s_n = a^n + b^n + c^n$, namely

$$s_0 = 3, \quad s_1 = 0 \quad \text{and} \quad s_2 = 6$$

and for $n \geq 3$,

$$s_n = 3s_{n-2} - s_{n-3}.$$

Calculating, we have

$$s_3 = -3, \quad s_4 = 18 \quad \text{and} \quad s_5 = -15.$$

Then the coefficient of x^2 is 15.

The last term is a bit harder. We can use the identity

$$a^5b^5 + b^5c^5 + c^5a^5 = \frac{1}{2}(s_5^2 - s_{10}).$$

And now we are left with the task of finding s_{10}. One can verify that

$$s_6 = 57, \ s_7 = -63, \ s_8 = 186, \ s_{10} = 621.$$

And therefore

$$a^5b^5 + b^5c^5 + c^5a^5 = \frac{1}{2}(s_5^2 - s_{10}) = -198.$$

So

$$Q(x) = x^3 + 15x^2 - 198x + 1.$$

5. Let a, b and c denote the lengths of the sides of $\triangle ABC$. The given equality of the areas of $\triangle ABC$ and $\triangle AB_2C_2$ implies

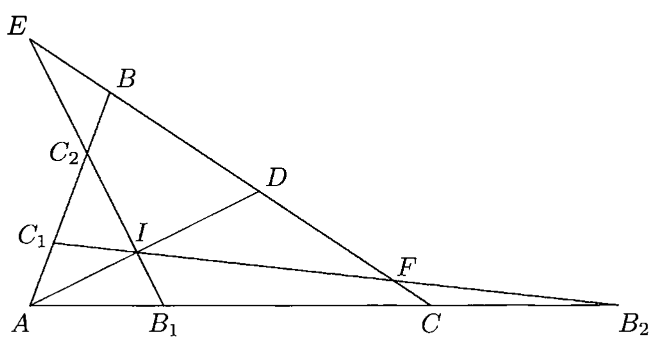
$$bc = AB_2 \cdot AC_2.$$

Then either

$$AB_2 \geq b, \quad AC_2 \leq c \quad \text{or} \quad AB_2 \leq b, \quad AC_2 \geq c.$$

Without loss of generality, we shall assume that the former holds. It follows that $b \geq a \geq c$.

Let D, E and F be the intersection points of the line BC with AI, B_1I and C_1I respectively. Then F lies in the interior of the segment BC, while E lies on its exterior.



A well known result in triangles states that

$$\begin{aligned} BD &= \frac{ac}{b+c} \\ CD &= \frac{ab}{b+c} \\ \text{and} \quad \frac{AI}{ID} &= \frac{b+c}{a}. \end{aligned}$$

By Menelaus' theorem applied on the triangle $\triangle ADB$ and the line C_1IF,

$$\frac{BF}{FD} = \frac{C_1B}{C_1A} \cdot \frac{IA}{ID} = \frac{b+c}{a}.$$

Therefore

$$\frac{BD}{FD} = \frac{b+c-a}{a}$$

and we get

$$FD = \frac{a^2c}{(b+c)(b+c-a)}$$

and

$$FC = CD - FD = \frac{a(b-a)}{b+c-a}.$$

Applying the theorem of Menelaus on the triangle $\triangle ABC$ with the line B_2FC_1 yields

$$\frac{B_2C}{B_2A} = \frac{FC}{FB} = \frac{b-a}{c}.$$

Simplifying, we get

$$B_2A = \frac{bc}{a+c-b}.$$

In the same manner we see that

$$C_2A = \frac{bc}{a+b-c}.$$

We recall the equality $AB_2 \cdot AC_2 = bc$, which becomes now

$$bc = \frac{b^2c^2}{(a+(b-c))(a-(b-c))} \Rightarrow a^2 = b^2 + c^2 - bc.$$

By the cosine law, this implies $\angle BAC = 60^\circ$.

6. We shall generalize the result of the problem in the following manner:

Let k be a positive integer, $S = 12k$, $m = 3k+2$. If

$$a_1 \leq a_2 \leq \cdots \leq a_m \leq 6k$$

are positive integers which sum up to S, then there exists a subset which sums to $S/2 = 6k$.

Recall that a sequence is called *universal* if every positive integer between 1 and the sum of the sequence can be expressed as a sum of some terms from the sequence. One can show by induction that if the sequence $x_1 \leq x_2 \leq \ldots \leq x_n$ satisfies

$$x_i \leq 1 + \sum_{j=1}^{i-1} x_j$$

then it is universal. In particular, if $x_i \leq i$ then it is universal.

We begin the proof by showing that in general, if $S < 4m-6$ (which holds in our case, since $S = 4m-8$) then for every $3 \leq i \leq m-2$, $a_i \leq i$. Indeed, for every i,

$$S = \sum_{j=1}^{i-1} a_j + \sum_{j=i}^{m} a_j \geq (i-1) + (m+1-i)a_i.$$

Hence

$$a_i \leq \frac{S+1-i}{m+1-i} = 1 + \frac{S-m}{m+1-i}.$$

We shall show that

$$\frac{S-m}{m+1-i} < i,$$

and the claim will follow. The last inequality is equivalent to

$$i^2 - (m+1)i + (S-m) < 0.$$

Since for $i = 3, i = m-2$ the expression on the left is negative (recall that $S < 4m-6$) then for i between those values, $a_i \leq i$.

If the sequence a_i is universal from the beginning, i.e $a_i \leq i$ for every $1 \leq i \leq m-2$. If

$$\sum_{i=1}^{m-2} a_i \geq S/2$$

then since the sequence is universal, there exists a subset of the sequence which sums to $S/2$. Otherwise, $a_{m-1} + a_m \geq S/2$ so $S/4 \leq a_m \leq S/2$, and since clearly

$$\sum_{i=1}^{m-2} a_i \geq S/4 = m-2$$

then $S/2$ can be expressed as the sum of a_m and a subset of $a_1, a_2 \ldots a_{m-2}$.

Therefore we shall assume that the first two terms of the sequence do not join into a universal sequence. It follows that $a_1 + a_2 \geq 4$, and $a_3 \geq 2$. Consider two cases:

Case 1: $a_3 = 2$. Then $a_1 = a_2 = a_3 = 2$. Let $2d$ be the number of odd terms of the sequence (it is even, since the sum is even). Then $2d \leq m-3$. We shall compose a new sequence $b_1 \leq b_2 \leq \ldots \leq b_{m-d}$ whose terms are half the even terms of a_i or the average of two odd terms. Then $b_1 = b_2 = b_3 = 1$,

$$\sum_{i=1}^{m-d} b_i = S/2$$

and since $a_m + a_{m-1} \leq S/2$ (for otherwise there should have been an element a_i smaller than 2), $b_{m-d} \leq S/4$. Since $S/2 < 4(m-d)-6$ we can apply the first claim and get that the sequence b_i is universal from the beginning and hence $S/4$ can be expressed as a sum of terms from this sequence. Therefore $S/2$ can be expressed as a sum of terms from the sequence a_i.

Case 2: $a_3 = 3$. We can refine the inequality for a_i from the beginning of the proof:

$$S = a_1 + a_2 + \sum_{j=3}^{i-1} a_j + \sum_{j=i}^{m} a_j \geq 4 + 3(i-3) + (m+1-i)a_i.$$

Therefore

$$a_i \leq \frac{S+5-3i}{m+1-i} = 3 + \frac{m-6}{m+1-i}.$$

For $i \leq 6$, the last expression is less than 4 and therefore $a_3 = a_4 = a_5 = a_6 = 3$. For $i \leq 2k+4$, it follows that $a_i < 6$ and hence the elements $a_3, a_4, \ldots a_{2k+4}$ are equal to either 3, 4 or 5. Let t be the number of elements from the last sequence that are equal to 3, and let $s = 2k+2-t$ be the number of elements that are equal to 4 or 5. Then $t \geq 4$ and if $t \geq 2k$ we are through. We can assume therefore that $t < 2k$ or $s \geq 3$.

For the rest of the proof, we need only those $2k+2$ elements. Let $3h$ be the largest multiple of 3 that can be expressed as the sum of the s 4s and 5s. By induction on s, starting from $s=3$ it follows that $h \geq s-1$, and that either 9,12,or 15 is the least multiple of 3 that can be expressed in this way. If $h \leq 2k$, so $3h \leq S/2$ then we can add multiples of 3 until we reach $S/2$ (since

$$3h + 3t \geq 3(s+t) - 3 = S/2 + 3).$$

Otherwise, remove the multiples of 3 from the sum of the 4s and 5s until we decrease under $S/2$. Since the least possible multiple of 3 we can remove is always at most 15, then at the end we reach at least $S/2 - 12$. Add the necessary number of 3s (recall that $t \geq 4$) and we are done.

5. TEAM COMPETITION PROBLEMS

1990

1. Let α be a positive rational number. Show that there exists an open interval I which contains α and for any rational $\beta \in I$, $\beta \neq \alpha$, the denominator of β is greater than that of α (the fractions should be considered in their simplest form).

2. For any rational α denote by I_α the maximal interval with the above property.

Find I_α if $\alpha = \dfrac{19}{90}$.

3. Let a and b be real numbers for which $0 < a < b < a + 1$. Prove that there exists such a rational number α, for which $a < \alpha < b$ and $a, b \subseteq I_\alpha$, i.e., for any rational β for which $a < \beta < b$ and $\beta \neq \alpha$, the denominator of β is greater than the denominator of α.

4. For any positive real pair $(a; b)$ denote the above α by $\alpha(a; b)$. Find the values of

$$\alpha\left(\frac{70}{177}; \frac{27}{68}\right).$$

$$\alpha\left(\sqrt{1990}; \sqrt{1991}\right).$$

Devise an algorithm as quick as possible to calculate the value of α.

1991

1. Let $f(x) = x^2 + ax + b$, where a, b are integers. Prove that if $f(x)$ is a square number for infinitely many integer values of x then $f(x)$ is the square of some integer polynomial.

2. Show that there exists an integer polynomial $f(x) = ax^2 + bx + c$, which is not a perfect square and $f(x)$ is a square number for infinitely many integer values of x.

3. Show that if $N > 0$ is an arbitrary integer then there exists an integer polynomial $f(x) = x^2 + ax + b$, which is not a perfect square and $f(x)$ is a square number for at least N integer values of x.

4. If $f(x) = x^2 + ax + b$ is an integer polynomial which is not a perfect square and f assumes a square value for N consecutive integer values of x, then f is called an N-square polynomial. Denote the discriminant $a^2 - 4b$ of $f(x)$ by $D(f)$.

   (a) Prove that if f is an N-square polynomial ($N > 2$) then $64 \mid D(f)$.

   (b) The square values of an N-square polynomial are alternately even and odd.

5. (a) Construct a 3-square polynomial.

   (b) Construct a 4-square polynomial.

1992

Preface

Consider the following two sequences:

1. The Fibonacci sequence is defined as
$$F_0 = 0, \qquad F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$
2. The definition of the Lucas-numbers is:
$$L_0 = 2, \qquad L_1 = 1, \qquad L_n = L_{n-1} + L_{n-2} \quad (n \geq 2).$$

It is well known, that for any $n \geq 0$,
$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$
where
$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$
You can use the results stated above, however, any other property of these sequences which you want to use should be proved.

The problems

1. Prove that $1 + L_{2^j} \equiv 0 \pmod{2^{j+1}} \quad (j \geq 0)$
2. Prove that $\displaystyle\sum_{k=1}^{n} \left[\alpha^k F_k + \frac{1}{2}\right] = F_{2n+1} \quad (n \geq 1).$
3. A natural number is called r-Fibonacci, if it can be written as the sum of r — not necessarily distinct — Fibonacci numbers ($r \geq 1$).

   Prove that there are infinitely many numbers which are not r-Fibonacci for any r $(1 \leq r \leq 5)$.
4. Prove that
$$F_n \cdot F_{n-1} \cdot F_n \cdot L_n \cdot L_{n-1} \cdot L_{n+1} \quad (n \geq 2)$$
is not a perfect square.
5. Prove that
$$L_{n+1} + (-1)^{n+1} \quad (n \geq 1)$$
can be written as the product of three (not necessarily distinct) Fibonacci numbers.
6. The coordinates of the vertices of a rectangle are all Fibonacci numbers. It is also given that there are no vertices of this rectangle on the coordinate axes. Prove that the sides of this rectangle are either parallel to the axes or they make a 45° angle with them.

1993

1. Let $k > 1$ such that for every $x, y \in G$

$$(xy)^i = x^i y^i \text{ holds for } i = k-1, k \text{ and } k+1.$$

Prove that G is Abelian.

2. Let $n \geq 1$ such that the mapping $x \to x^n$, $x \in G$ is an isomorphism of G onto itself. Show that $a^{n-1} \in G$ for every $a \in G$.

3. Prove that every element of S_n can be expressed as the product of two cycles.

4. Let $H \leq G$, $a, b \in G$. Prove that $|aH \cap Hb|$ is either zero or a divisor of $|H|$.

5. Let $|H| = 3$ $(H \leq G)$. What can be said about $|N_G(H) : C_G(H)|$?

6. Let $a, b \in G$. Assume that

$$ab^2 = b^3 a, \quad ba^2 = a^3 b.$$

Prove that $a = b = 1$.

7. Let $|G'| = 2$. Prove that $|G : G'|$ is even.

GLOSSARY of Notations

G:	a finite group
G':	the commutator subgroup of G
$H \leq G$:	H is a subgroup of G
$N_G(H)$:	the normalizer of H in G.
$\lvert G : H\rvert$:	the index of the subgroup H in G.
$C_G(H)$:	the centralizer of H in G.
$\lvert X\rvert$:	the cardinality of the subset $X \subset G$.
S_n:	the symmetric group of degree n.
$Z(G)$:	the centre of G.

1994

1. Let $G(V, E)$ be an undirected graph, V is the set of vertices, E is the set of edges. Let $w : E \to R$ be a function assigning a real number called its weight to each edge. The *max-weight* of a non-empty set of edges is the maximum of the weights of the edges contained in the set. Suppose that G is connected and denote $|E|$ by m. Find a spanning tree in $O(m)$ steps whose max-weight is minimal (in one step two real numbers can be compared and the corresponding pointer administration can also be performed).

2. $G(V, E)$ is a connected undirected graph, $w : E \to R$ is a weighing function and $e(x, y)$ is a given edge of G. Decide in $O(m)$ steps ($m = |E|$) if there exists a minimal spanning tree — whose weight is minimal — containing e.

   Remark: Graphs are always given by the corresponding adjacency list, i.e. for every vertex we are given the list of its neighbours.

3. Given a $G(V, E)$ directed graph, one of its vertices s is fixed as the "source", a function $w : E \to R$ assigning a positive weight for every edge and another function $d : E \to R$. Someone claims that $d(x)$ is the minimum of the weights of the directed paths $s \to x$ for every vertex $x \in V$.

   Decide in $O(n + m)$ steps if she is right ($n = |V|$, $m = |E|$; in one step you can perform an arithmetic operation or the comparison of two real numbers with the corresponding pointer administration).

4. An undirected graph is k-regular if the degree of every edge is k.

   (a) Show that the set of edges of any 3-regular graph $G(V, E)$ can be split into two subsets E_1, E_2 such that the degree of any point is at most 2 in both of the graphs (V, E_1) and (V, E_2).

   (b) Find such a division E_1, E_2 in $O(m)$ steps ($m = |E|$).

   (c) A colouring of the edges of a graph is "good" if the colour of the edges having a common vertex is different. Find an algorithm in $O(m)$ steps which well-colours the edges of a 3-regular graph with 4 colours.

5. $x_1, x_2, \ldots, x_n \in \{+1, -1\}$ are unknown numbers. We have to find the value of $\prod x_i$, i.e. the parity of the number of -1 elements of the above list.

   We can ask "linear" questions. A linear question is an ordered list $a_1, a_2, \ldots, a_n, b$ of $n + 1$ real numbers. The answer for such a

question is YES if

$$a_1x_1 + a_2x_2 + \cdots + a_n > b$$

and NO otherwise.

Prove that if we have a strategy yielding the value of $\prod x_i$ using at most k linear questions (independently of the values of $x_1, x_2, \ldots, x_n$), then $k > \log_2 n$.

Remark: Our strategy is deterministic: each question is a function of the previous replies.

1995

1. Consider the family of curves whose equation is $x^4 - x^2 = a$, where a is an arbitrary positive number. Cut these curves with a line l, which is parallel to the y-axis. Draw the tangent to each curve at the point of intersection.

   (a) Prove that these tangent lines are passing through a common point P_l.

   (b) Find the locus of the points P_l as the line l assumes every position parallel to the y-axis.

2. P is a point on the curve $x^{2/3} + y^{2/3} = 1$ $(x, y > 0)$ and the tangent to the curve at P cuts the y-axis at the point Q. R is a point on this tangent such that its first coordinate is not negative and $QR = b$, where b is a given positive number. What is the locus of the points R as P is moving along the curve?

3. Consider the curve C in the space with parametric equation

$$\{t, t^2, t^3 \mid t \in R\}.$$

   Find the points $P(a, b, c)$ such that projecting C from P to the x, y-plane the resulting cubic curve has a singularity, with

   (a) a cusp,

   (b) distinct tangent directions (the curve cuts itself).

1996

1. The graph G has 20 vertices, it is triangle-free and the degree of each vertex is a multiple of 4. What is the maximum number of edges in G?

2. G is a graph of n vertices ($n > 1$). Suppose that the degree of each vertex is at least $(n-1)/2$. Prove that there exists a set S of vertices such that the size of S is less than $1 + \log_2 n$ and each vertex outside S is connected to at least one vertex in S by an edge.

3. For which values of k does every connected k-regular bipartite graph contain a Hamiltonian circuit?

4. s and t are distinct vertices of a directed graph. Assume that the in-degree of each vertex is equal to its out-degree and, furthermore, for each subset S of the vertices containing t but not s, the number of edges entering S is at least k. Prove that there exist $2k$ edge-disjoint paths connecting s and t such that half of them are from s to t while the other half are from t to s.

5. Which one has the biggest number of edges among those graphs of 1000 vertices which do not contain a path of length 100?

6. Denote the maximum number of edges in those connected graphs which have k vertices and do not contain a path of length 100 by $f(k)$.

   Prove that there exist constants c_1, c_2 such that

   $$49k - c_1 \leq f(k) \leq 49k - c_2.$$

   Find as good estimates for the two constants c_1 and c_2 for

   i) every value of k;

   ii) k sufficiently large.

2000

1. Let $\triangle ABC$ be a given triangle. P_1 is a point inside $\triangle ABC$.

(a) Prove that the lines obtained by reflecting P_1A, P_1B, P_1C through the angle bisectors of $\angle A$, $\angle B$, $\angle C$, respectively, meet at a common point P_2.

(b) Let A_1, B_1, C_1 be the feet of the perpendiculars from P_1 on BC, CA and AB, respectively. Let A_2, B_2, C_2 be the feet of the perpendiculars from P_2 on BC, CA and AB, respectively. Prove that these six points A_1, B_1, C_1, A_2, B_2, C_2 lie on a circle.

(c) Prove that the circle of part (b) touches the nine point circle (*Feuerbach's circle*) of $\triangle ABC$ if and only if P_1, P_2 and the centre of the circumcircle of $\triangle ABC$ are collinear.

2. An ant is walking inside the region bounded by the curve whose equation is

$$x^2 + y^2 + xy = 6.$$

Its path is formed by straight segments parallel to the coordinate axes. The ant starts at an arbitrary point on the curve and takes off inside the region. When reaching the boundary, it turns by 90° and continues its walk inside the region. When arriving at a point on the boundary which it has already visited, or where it cannot continue its walk according to the given rule, the ant stops. Prove that, sooner or later, and regardless of the starting point, the ant will stop.

3. (a) In the plane, we are given the circle C (without its centre) and the point P. Is it possible to construct, with a ruler only, the line through P and the centre of the circle?

(b) In the plane, we are given two circles C_1 and C_2 (without their centres). Construct, with a ruler only, the line through their centres when:

i. the two circles intersect.

ii. the two circles touch each other, and their point of contact, T, is marked.

iii. $*$ *We do not know how to solve this problem if the two circles have no common point. It might even be the case that this construction cannot be done at all with a ruler only. Can you do better?*

Note: Part (iii) does not belong to the official team contest. However, any progress will be appreciated.

2001

In the following questions, G_n is a simple undirected graph with n vertices, K_n is the complete graph with n vertices, $K_{n,m}$ is the complete bipartite graph with m vertices on one party and n vertices on the other party, and C_n is a circle with n vertices. $e(G_n)$ is the number of edges in the graph G_n.

1. The edges of K_n, $n \geq 3$ are coloured with n colours, and every colour appears at least once. Prove that one can find a triangle whose sides are coloured with 3 different colours.

2. $n \geq 5$ is given. If $e(G_n) \geq \frac{n^2}{4} + 2$, prove that there exist two triangles which has exactly one common vertex.

3. $e(G_n) \geq \frac{n\sqrt{n}}{2} + \frac{n}{4}$. Prove that G_n contains C_4.

4. (a) G_n does not contain $K_{2,3}$. Prove that $e(G_n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + n$.

   (b) Given $n \geq 16$ distinct points $P_1, P_2, \ldots, P_n$ in the plane, prove that at most $n\sqrt{n}$ of the segments P_iP_j has unit length.

5. (a) Let p be a prime. Consider the set $\{(x,y) | 0 \leq x, y \leq p-1\}$ of vertices, such that $(x,y), (x',y')$ are connected if $xx' + yy' \equiv 1 \pmod{p}$. Prove that this graph does not contain C_4.

   (b) Prove that for infinitely many values of n, there exists a graph G_n that does not contain C_4 and $e(G_n) \geq \frac{n\sqrt{n}}{2} - n$.

6. TEAM COMPETITION SOLUTIONS

1990

1. Let $\alpha = \dfrac{m}{n}$ with $\gcd(m, n) = 1$.

Denote by S the set of rationals with denominator not greater than n. The set S is well-ordered, and hence there exist rationals $\beta, \gamma \in S$ such that β is the maximal element in S smaller than α, and γ is the minimal element in S greater than α. Then in the interval $I = (\beta, \gamma)$ there are no rationals with denominator less than or equal to n, except α.

2. Let S_n be set the of rationals with denominator less than or equal to n. As we stated, S_n is well-ordered for every n. The Farey sequence

$$\left(F_n^{(m)}\right)_{m=1}^{\infty}$$

is defined to be the elements in S_n with this order. Clearly, if $\alpha = F_n^{(k)}$, then

$$I_\alpha = (F_n^{(k-1)}, F_n^{(k+1)}).$$

We shall now prove the following property of the Farey sequence:

If

$$F_n^{(m)} = \frac{a}{b}, \quad F_n^{(m+1)} = \frac{c}{d}$$

with

$$\gcd(a, b) = \gcd(c, d) = 1$$

then $bc - ad = 1$.

Proof: Since $F_n^{(m)} < F_n^{(m+1)}$ then

$$\frac{a}{b} < \frac{c}{d} \Longrightarrow bc - ad > 0 \Longrightarrow bc - ad \geq 1.$$

We shall now prove that if a, b natural numbers with $\gcd(a, b) = 1$ then there exist natural numbers c, d such that $d \leq b$, $\gcd(c, d) = 1$ and $bc - ad = 1$, which will complete the proof.

To that end, consider the set $\{ax | 1 \leq x \leq b\}$. If for $1 \leq d \leq b$, $ad \equiv 1 \pmod b$ then we are through. Otherwise, since there are b different residues modulo b, it follows from the pigeonhole principle that for $1 \leq x_1 < x_2 \leq b$,

$$ax_1 \equiv ax_2 \pmod b.$$

Then $b|a(x_2-x_1)$, and since $\gcd(a,b)=1$, $b|x_2-x_1$, but $b>x_2-x_1$ and we arrive at a contradiction.

It remains to find rationals $\beta=\frac{a}{b}$, $\gamma=\frac{c}{d}$ such that $19b-90a=1$, $90c-19d=1$ and $b,d\le 90$.

$$\begin{aligned} 19b &= 90a+1 \Rightarrow 1-5a\equiv 0 \pmod{19} \Rightarrow a=4 \Rightarrow \beta=\frac{4}{19} \\ 90c &= 19b+1 \Rightarrow 1+5c\equiv 0 \pmod{19} \Rightarrow c=15 \Rightarrow \gamma=\frac{15}{71} \end{aligned}$$

Therefore

$$I_{\frac{19}{90}}=(\frac{4}{19},\frac{15}{71}).$$

3. It is a well-known fact that between any two real numbers lies a rational number.

Let $a<\varepsilon<b$, $\varepsilon=\frac{u}{v}$. By the minimum principle, there exists a rational $\alpha\in(a,b)$ such that the denominator of α is minimal.

We shall now prove that α is unique. Let $\alpha=\frac{r}{s}$, and assume that

$$\beta=\frac{r\pm 1}{s}\in(a,b).$$

Then since

$$(r\pm 1)s-sr=\pm s\neq 1,$$

it follows that α,β aren't consecutive in F_s, so there exists $\gamma\in(a,b)$ such that γ is between α and β in F_s, and thus the denominator of γ is less then s, contradicting the minimality of s.

Therefore, α is unique.

4. Of course, there is the algorithm of finding the consecutive elements in the Farey sequence, as we used in the previous problem. We shall use now a different method, consisting of the properties of continued fractions.

For the first case, note that

$$a=\frac{70}{177}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{4}}}}},\qquad b=\frac{27}{68}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{13}}}}.$$

Now

$$\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8}}}} = \frac{17}{43} < \frac{70}{177}, \qquad \cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}} = \frac{2}{5} > \frac{27}{68}$$

and

$$\frac{2}{5} - \frac{17}{43} = \frac{1}{5 \cdot 43}.$$

Therefore $\frac{2}{5}, \frac{17}{43}$ are consecutive elements in F_{43}. Hence the number with the least denominator between them is

$$\frac{17+2}{43+5} = \frac{19}{48}.$$

Since

$$\frac{19}{48} - \frac{17}{43} = \frac{1}{43 \cdot 48}, \qquad \frac{2}{5} - \frac{19}{48} = \frac{1}{5 \cdot 48}.$$

Then $\frac{17}{43}$, $\frac{19}{48}$ and $\frac{2}{5}$ are consecutive elements in F_{48}. This leads us to the conclusion that $\frac{19}{48}$ is the number with the least denominator in the interval $(\frac{17}{43}, \frac{2}{5})$ and thus

$$\alpha\left(\frac{70}{177}; \frac{27}{68}\right) = \frac{19}{48}.$$

For the second case, note that the representations of $\sqrt{1990}$ and $\sqrt{1991}$ as continued fractions are

$$\sqrt{1990} = 44 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{\ddots}}}}}}$$

and $$\sqrt{1991} = 44 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{\ddots}}}}}}}$$

Let

$$x = 44 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}}} = 44 + \frac{5}{8}$$

and

$$y = 44 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{1}}}} = 44 + \frac{3}{5}.$$

We have $x < \sqrt{1990} < \sqrt{1991} < y$, and $y - x = \frac{1}{40}$ and hence x, y are consecutive elements in F_8. Therefore

$$\frac{3}{5}, \quad \frac{3+5}{5+8} = \frac{8}{13}, \quad \text{and} \quad \frac{5}{8}$$

are consecutive in F_{13} because

$$\frac{8}{13} - \frac{3}{5} = \frac{1}{65}$$

and

$$\frac{5}{8} - \frac{8}{13} = \frac{1}{104}.$$

Finally, we get

$$\alpha\left(\sqrt{1990}; \sqrt{1991}\right) = 44 + \frac{8}{13}.$$

1991

1. Assume that the integer polynomial

$$f(x) = x^2 + ax + b$$

is a square number for infinitely many values of x. If

$$x^2 + ax + b = y^2$$

then the discriminant

$$a^2 - 4b + 4y^2$$

must be a perfect square, say

$$z^2 = a^2 - 4b + 4y^2.$$

There are therefore infinitely many values of y and z such that

$$z^2 - (2y)^2 = a^2 - 4b.$$

Clearly, the values of z^2 and y^2 increase beyond all bounds, and so do the values of z, y.

Assume by contradiction that

$$a^2 - 4b = n \neq 0.$$

Then

$$(z - 2y)(z + 2y) = n,$$

but $z + 2y$ increases beyond all bounds, while $z - 2y$ is at least 1, contradicting n is finite.

Therefore, $a^2 = 4b$, and there exists an integer c such that $a = 2c$ and $b = c^2$. Thus

$$f(x) = x^2 + 2cx + c^2 = (x + c)^2.$$

2. Let

$$f(x) = 2x^2 + 3x + 1,$$

for all integers y, z such that

$$z^2 - 8y^2 = 1.$$

Then if $z \equiv 3 \pmod 4$, we have

$$f\left(\frac{z-3}{4}\right) = 2 \cdot \frac{z^2 - 6z + 9}{16} + 3 \cdot \frac{z-3}{4} + 1 = \frac{z^2 - 1}{8} = y^2,$$

and if $z \equiv 1 \pmod 4$ then similar calculation yields

$$f\left(\frac{-z-3}{4}\right) = y^2.$$

It is left to prove that there are indeed infinitely many solutions to the equation $z^2 - 8y^2 = 1$ in integers.

This equation is a particular case of a more general case of Pell's equation. A first solution is provided by $3^2 - 8 \cdot 1^2 = 1$, and every other solution is given by

$$x + y\sqrt{8} = (3 + \sqrt{8})^n.$$

Obviously, there are infinitely many solutions and thus $f(x)$ is a perfect square for infinitely many values of x.

3. For every N, consider the polynomial

$$f(x) = x^2 + (2^N + 4) + (2^{N+1} + 4).$$

If y, z are integers such that

$$z^2 - 4y^2 = 2^{2N} = a^2 - 4b$$

and z is even, then

$$f\left(\frac{-2^N - 4 \pm z}{2}\right) = y^2.$$

We shall prove that there are $2N - 1$ different solutions to the equation

$$z^2 - 4y^2 = 2^{2N},$$

yielding $2N - 1$ values of x for which $f(x)$ is a perfect square.

For every $k \neq \pm 1$, then the system of equations

$$\begin{cases} z - 2y = & 2^k \\ z + 2y = & 2^{2N-k} \end{cases}$$

has the solution

$$(y, z) = \left(2^{n-k-2} - 2^{k-2}, 2^{k-1} + 2^{n-k-1}\right),$$

with z even.

Since there are $2N - 1$ different options for k, it follows that here are $2N - 1$ solutions to that equation, thus $2N - 1$ values of x such that $f(x)$ is a perfect square.

4. Let us prove part (b) first. Denote the smallest x for which $f(x)$ is a perfect square by x_0, let $y_0 = \sqrt{f(x_0)}$ and let z_0 be the discriminant

$$\sqrt{a^2 - 4b + 4y_0^2}.$$

For every $0 \le k < N$, we have $f(x_0 + k) = y_k^2$, and thus

$$z_k = 2(x_0 + k) + a = z_0 + 2k.$$

We have

$$D(f) = z_0^2 - (2y_0)^2 = (z_0 + 2)^2 - (2y_1)^2 = (z_0 + 4)^2 - (2y_2)^2,$$

implying

$$2D(f) = 2(z_0 + 2)^2 - 2(2y_1^2) = z_0^2 + (z_0 + 4)^2 - (2y_0)^2 - (2y_2)^2.$$

Therefore

$$y_0^2 + y_2^2 = 2y_1^2 + 2.$$

Checking residues modulo 4, we find that the values of y_0, y_1, y_2 can be *odd*, *even*, *odd* or *even*, *odd*, *even* respectively. Obviously, this relation holds for every y_k, y_{k+1}, y_{k+2} when $0 \le k < N - 2$. Therefore, the values of

$$y_k = \sqrt{f(x_k)}$$

are alternately odd and even.

For part (a), note that $z_0 + 1 = y_1^2 - y_0^2$. Since the parity of y_0, y_1 is different, z_0 is even. Let us check residues modulo 8:

$$\begin{aligned}
z_0 \equiv 0 \pmod 8 :\quad & y_1^2 - y_0^2 \equiv 1 \pmod 8\\
& \Rightarrow y_0 \equiv 0 \pmod 4\\
& \Rightarrow D(f) = z_0^2 - (2y_0)^2 \equiv 0 \pmod{64}\\
z_0 \equiv 2 \pmod 8 :\quad & y_1^2 - y_0^2 \equiv 3 \pmod 8\\
& \Rightarrow y_1 \equiv 2 \pmod 4\\
& \Rightarrow D(f) = (z_0 + 2 + 2y_1)(z_0 + 2 - 2y_1)\\
& \qquad\quad \equiv 0 \pmod{64}\\
z_0 \equiv 4 \pmod 8 :\quad & y_1^2 - y_0^2 \equiv 5 \pmod 8\\
& \Rightarrow y_0 \equiv 2 \pmod 4\\
& \Rightarrow D(f) = (z_0 + 2y_0)(z_0 - 2y_0)\\
& \qquad\quad \equiv 0 \pmod{64}\\
z_0 \equiv 6 \pmod 8 :\quad & y_1^2 - y_0^2 \equiv 7 \pmod 8\\
& \Rightarrow y_1 \equiv 0 \pmod 4\\
& \Rightarrow D(f) = (z_0 + 2)^2 - (2y_0)^2\\
& \qquad\quad \equiv 0 \pmod{64}
\end{aligned}$$

Our claim is proved.

5. (a) We need to find three integers y_0, y_1, y_2 such that

$$2y_1^2 + 2 = y_0^2 + y_2^2$$

but $y_1 - y_0 \neq 1$, for otherwise $D(f) = 0$, implying f is a square polynomial.

Hence we have to solve the Pell equation

$$y_2^2 - 2y_1^2 = 2 - y_0^2.$$

Putting $y_0 = 1$, we have to solve the unit equation

$$u^2 - 2v^2 = 1.$$

The fundamental solution of this equation is $(3, 2)$, and all other solutions are given by

$$u + v\sqrt{2} = (3 + 2\sqrt{2})^n.$$

Taking $n = 2$, we have

$$y_2 + y_1\sqrt{2} = 17 + 12\sqrt{2},$$

implying $y_2 = 17, y_1 = 12$. Then

$$z_0 = 12^2 - 1 - 1 = 142$$

and thus $D(f) = 20160$. Therefore we can choose $a = b = 144$, and thus

$$f(x) = x^2 + 144x + 144$$

receives square values for $x = -1, 0, 1$.

(b) We shall show a different method of finding an N-polynomial. Suppose that $D = pq$ with

$$\sqrt{k} < \sqrt{p} - \sqrt{q} < \sqrt{k+1}.$$

Then

$$\begin{aligned} p + q - 1 &= (\sqrt{p} - \sqrt{q})^2 + 2\sqrt{pq} \\ &< k + 2\sqrt{D} \\ &< (\sqrt{p} - \sqrt{q})^2 + 2\sqrt{pq} \\ &= p + q. \end{aligned}$$

Hence

$$\lceil 2\sqrt{D} + k \rceil = p + q,$$

and it follows that

$$\lceil 2\sqrt{D}+k\rceil^2 - 4D = (p-q)^2.$$

Note now that if D is given then there can be only one pair (p,q) such that

$$\sqrt{k} < \sqrt{p} - \sqrt{q} < \sqrt{k+1}$$

for a given k, since if (p,q) are a pair satisfying this inequality, then pq and $p+q$ are determined uniquely.

We can now present a way of finding an N-polynomial. We need to find an integer D for which there are N pairs p_i, q_i such that $D = p_i q_i$ and

$$0 < \sqrt{p_i} - \sqrt{q_i} < \sqrt{N}.$$

If we have found such D, then the polynomial $f(x) = x^2 - 4D$ is a perfect square for

$$x = \lceil 2\sqrt{D} + i\rceil, \qquad i = 0, 1, \ldots N-1.$$

In order to find a 4-polynomial, we have to find D such that $D = pq$ and

$$0 < \sqrt{p} - \sqrt{q} < 2$$

holds for four different pairs p, q. One solution is $D = 15120$ with the pairs $(126,120)$, $(135,112)$, $(140,108)$, $(144,105)$. Then

$$\lceil 2\sqrt{15120}\rceil = 246$$

and

$$f(x) = x^2 - 4D = x^2 - 60480$$

is a perfect square for $x = 246$, 247, 248 and 249.

1992

1. We shall use induction on j. The case $j = 0$ is trivial. Assume that the claim holds for j, that is, $L_{2^j} + 1 = m \cdot 2^{j+1}$. Then

$$\begin{aligned} L_{2^{j+1}} &= \alpha^{2^{j+1}} + \beta^{2^{j+1}} \\ &= \left(\alpha^{2^j} + \beta^{2^j}\right) - 2(\alpha\beta)^{2^j} \\ &= (m \cdot 2^{j+1} - 1) - 2 \\ &= 2^{j+2}(m^2 \cdot 2^j - m) - 1 \\ &\equiv -1 \pmod{2^{j+2}}. \end{aligned}$$

Which is what we wished to prove.

2. We will prove that for every k,

$$\left\lfloor \alpha_k F_k + \frac{1}{2} \right\rfloor = F_{2k}.$$

This is equivalent to

$$\frac{\alpha^{2k} - \beta^{2k}}{\sqrt{5}} \leq \frac{\alpha^k - \beta^k}{\sqrt{5}} \cdot \alpha^k + \frac{1}{2} < \frac{\alpha^{2k} - \beta^{2k}}{\sqrt{5}} + 1.$$

Simplifying, we get the inequalities

$$-\beta^{2k} \leq \frac{\sqrt{5}}{2} - (-1)^k < \sqrt{5} - \beta^{2k}.$$

We will prove the left inequality first. Since $\beta < 1$ it follows that

$$\frac{\sqrt{5}}{2} > 1 > (-1)^k > (-1)^k - \beta^{2k}.$$

For the second inequality, if $k \geq 3$, then

$$\sqrt{5} - \frac{\sqrt{5}}{2} + (-1)^k > \frac{\sqrt{5}}{2} - 1 > \beta^5 \geq \beta^{2k},$$

and it is left only to verify the inequality to $k = 0, 1, 2$.

This yields

$$\sum_{k=0}^{n} \left\lfloor \alpha^k F_k + \frac{1}{2} \right\rfloor = \sum_{k=0}^{n} F_{2k} = F_1 + \sum_{k=1}^{n} F_{2k} = F_{2n+1}.$$

3. First, any number which is r-Fibonacci, where $1 \leq r \leq 5$ is 5-Fibonacci as well, since we can add $F_0 = 0$ to the sum. Also, if

$$n = F_{k_1} + F_{k_2} + F_{k_3} + F_{k_4} + F_{k_5}$$

is the representation of n as 5-Fibonacci number, we will assume that $|k_i - k_j| \neq 1$.

Our claim is that for $k \geq 6$, $F_{2k+1} - 1$ is not 5-Fibonacci. Assume by contradiction that

$$F_{2k+1} - 1 = F_{k_1} + F_{k_2} + F_{k_3} + F_{k_4} + F_{k_5}$$

with $k_i + 1 < k_{i+1}$. Then

$$\begin{aligned} & F_{k_1} + F_{k_2} + F_{k_3} + F_{k_4} + F_{k_5} \\ \leq\ & F_{2k} + F_{2k-2} + F_{2k-4} + F_{2k-6} + F_{2k-8} \\ <\ & \sum_{i=0}^{2k} F_{2k} \\ =\ & F_{2k+1} - 1. \end{aligned}$$

A contradiction.

4. We note the following identities:

$$\begin{aligned} F_n L_n &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)(\alpha^n + \beta^n) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) \\ &= F_{2n} \\ F_{n-1} L_{n+1} &= \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1})(\alpha^{n+1} + \beta^{n+1}) \\ &= \frac{1}{\sqrt{5}}\left(\alpha^{2n} - \beta^{2n} - (\alpha\beta)^{n-1}(\alpha^2 - \beta^2)\right) \\ &= F_{2n} - (-1)^{n-1} \\ F_{n+1} L_{n-1} &= \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})(\alpha^{n-1} + \beta^{n-1}) \\ &= \frac{1}{\sqrt{5}}\left(\alpha^{2n} - \beta^{2n} + (\alpha\beta)^{n-1}(\alpha^2 - \beta^2)\right) \end{aligned}$$

$$= \quad F_{2n} + (-1)^{n-1}.$$

Therefore,

$$F_{n-1}F_nF_{n+1}L_{n-1}L_nL_{n+1} = F_{2n}(F_{2n}^2 - 1)$$

Suppose that $F_{2n}(F_{2n}^2 - 1)$ is a perfect square. Since

$$\gcd(F_{2n}, F_{2n}^2 - 1) = 1$$

it follows that both of them must be perfect squares, but since $F_{2n} > 1$, $F_{2n}^2 - 1$ cannot be a perfect square.

5. For every n, we claim that

$$L_{2n+1} + (-1)^{n+1} = F_5 \cdot F_n \cdot F_{n+1}.$$

The proof is very easy:

$$\begin{aligned} F_5 \cdot F_n \cdot F_{n+1} &= 5 \cdot \frac{\alpha^n - \beta^n}{\sqrt{5}} \cdot \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{5}} \\ &= \alpha^{2n+1} + \beta^{2n+1} - (\alpha+\beta)(\alpha\beta)^n \\ &= L_{2n+1} + (-1)^{n+1}. \end{aligned}$$

6. Denote the rectangle by $ABCD$ with

$$A = (F_{a_1}, F_{a_2}) \quad B = (F_{b_1}, F_{b_1}) C = (F_{c_1}, F_{c_2}) \quad D = (F_{d_1}, F_{d_2})$$

Since $ABCD$ is a rectangle, it follows that

$$F_{a_1} + F_{c_1} = F_{b_1} + F_{d_1} \quad \text{and} \quad F_{a_2} + F_{c_2} = F_{b_2} + F_{d_2}.$$

If, for example $F_{a_1} = F_{b_1}$ then $F_{c_1} = F_{d_1}$ and since $ABCD$ is a rectangle, we get $F_{a_2} = F_{b_2}$ and $F_{c_2} = F_{d_2}$ and thus the sides of $ABCD$ are parallel to the coordinate axes.

Therefore, assume that $F_{a_1} \neq F_{b_1}$ and $F_{a_1} \neq F_{d_1}$. With no loss of generality we can assume that $F_{d_1} \geq F_{a_1}, F_{b_1}, F_{c_1}$. Clearly, the inequality must be sharp, for otherwise we would get that F_{d_1} is equal to F_{c_1} or F_{a_1}. Suppose that $F_{a_1} \neq F_{c_1}$. Then since $F_{b_1} \neq 0$, it follows that

$$F_{a_1} + F_{c_1} \leq F_{d_1 - 1} + F_{d_1 - 2} < F_{d_1} + F_{b_1},$$

contradicting that $ABCD$ is a rectangle. Therefore, $F_{a_1} = F_{c_1}$.

In a similar manner, from the equality

$$F_{a_2} + F_{c_2} = F_{b_2} + F_{d_2},$$

we must have $F_{d_2} = F_{b_2}$ (since $F_{a_2} \neq F_{c_2}$), and therefore the diagonals of the rectangle $ABCD$ are parallel to the coordinate axes, and thus $ABCD$ is a square whose sides make $45°$ angles with the coordinate axes.

1993

1. We will use the following identity: for every $x, y \in G$ and for every $i > 1$,

$$(xy)^i = x \cdot (yx)^{i-1} \cdot y.$$

Let $x, y \in G$. It is given that

$$(xy)^{k+1} = x^{k+1}y^{k+1},$$

and using our identity we get

$$(yx)^k = x^k y^k = (xy)^k.$$

Similarly, we have

$$(yx)^{k-1} = (xy)^{k-1}.$$

But

$$(xy)^k = (xy) \cdot (xy)^{k-1}$$

and

$$(yx)^k = (yx) \cdot (yx)^{k-1},$$

which leads to the conclusion that $xy = yx$ for every $x, y \in G$, which is what we wished to prove.

2. Denote by φ the mapping $x \to x^n$. Since φ is an isomorphism, then for every $x, y \in G$ we have

$$\varphi(x) \cdot \varphi(y) = \varphi(xy),$$

that is,

$$x^n y^n = (xy)^n.$$

Let $a^{n-1} \in G$, and let b be an arbitrary element in G. The mapping φ is on G, and hence there exists $c \in G$ such that $b = c^n$. It follows that

$$\begin{aligned} a^{n-1}b &= a^{-1} \cdot (a^n \cdot c^n) \\ &= a^{-1} \cdot (ac)^n \\ &= a^{-1} \cdot \left(a \cdot (ca)^{n-1} \cdot c\right) \\ &= (ca)^n \cdot a^{-1} \\ &= c^n a^{n-1} \\ &= ba^{n-1}, \end{aligned}$$

and therefore $a^{n-1} \in Z(G)$ for every $a \in G$.

3. We shall prove the claim by using induction on n. Clearly, the claim holds for $n = 1, 2, 3, 4, 5$. Assume that for every $\sigma \in S_{n-1}$, we can write σ as a product of two cycles. Let $\sigma \in S_n$ with $n \geq 6$. Consider three cases:

Case 1: σ leaves one element in the set $\{1, 2, \ldots n\}$ in its place. Then there exist $\sigma' \in S_{n-1}$ such that $\sigma = \sigma'$, and thus by our induction hypothesis, σ is the product of two cycles.

Case 2: σ is a product of m cycles with size 2, where $n = 2m$, and without loss of generality we can assume

$$\sigma = (1\ 2)(3\ 4)\cdots(2m-1\ 2m).$$

Then

$$\sigma = (2m-1\ 2m-3\ \ldots 5\ 3\ 1)(1\ 2\ 3\ 4\ 5\ 6\ \ldots 2m)$$

Case 3: σ effects every element in $\{1, \ldots n\}$ and contains at least one cycle with length $l \geq 3$. Assume that

$$\sigma = (1\ 2\ \ldots l-1\ l) \cdot \tau,$$

where τ and the cycle are distinct (τ is not necessarily a cycle) and let

$$\bar{\sigma} = (1\ 2\ \ldots l-1) \cdot \tau.$$

By our induction, $\bar{\sigma}$ is the product of two cycles, say $\bar{\sigma} = \varphi \cdot \psi$. Since $\bar{\sigma}(l-1) = 1$, the cycle φ takes $\psi(l-1)$ to $\psi(1)$. We define a new cycle $\bar{\varphi}$:

$$\bar{\varphi}(k) = \begin{cases} \varphi(k) & k \neq \psi(l-1), l \\ l & k = \psi(l-1) \\ \psi(1) & k = l \end{cases}$$

It is easy to verify that $\bar{\varphi}$ is indeed a cycle, and that $\sigma = \bar{\varphi} \cdot \psi$.

4. Assume that

$$|aH \cap Hb| \neq 0,$$

then there exists an element of aH, say a, which can be written as $a = hb$ where $h \in H$. Denote by K the set $H \cap b^{-1}Hb$.

We shall prove that

$$|K| = |aH \cap Hb|.$$

Indeed, let $k \in K$, so that $k \in H$ and $k = b^{-1}h'b$, where $h' \in H$. Then $ak \in aH$, and

$$ak = hb \cdot b^{-1}h'b = hh'b \in Hb.$$

On the other hand, let $x \in aH \cap Hb$, so there exist $h_1, h_2 \in H$ such that

$$x = ah_1 = h_2 b.$$

This implies that

$$h_1 = a^{-1}h_2 b = b^{-1}h^{-1}h_2 b \in b^{-1}Hb.$$

We will now need to show that $|K|$ divides $|H|$.

To that end, recall that for every $b \in G$, $b^{-1}Hb$ is a subgroup of G. The intersection of two subgroups is a subgroup as well, so $K \leq G$. But $K \subseteq H$ and therefore $K \leq H$. By Lagrange's theorem, its order divides $|H|$.

5. The problem is solved easily using the "N/C theorem", which states that there exists a homomorphism

$$\varphi : N_G(H) \to Aut(H),$$

where $Aut(H)$ is the group of the automorphisms of H, such that $Ker(\varphi) = C_G(H)$. In other words, there exists a subgroup K of H for which

$$N_G(H)/C_G(H) \cong K.$$

Since $|H| = 3$, then

$$Aut(H) = S_2 \cong Z_2.$$

Using the N/C theorem, we deduce that $[N_G(H) : C_G(H)]$ divides 2, and therefore

$$[N_G(H) : C_G(H)] = 1 \text{ or } 2.$$

We shall now prove the N/C theorem. Define φ in the following manner: for every $g \in N_G(H)$,

$$(\varphi(g))(h) = ghg^{-1}.$$

Since $g \in N_G(H)$, this mapping is defined for every $h \in H$, and it is now easy to verify that the mapping $h \to ghg^{-1}$ is indeed an automorphism. The unit element in $Aut(H)$ is the identity function of H, which leads to

$$Ker(\varphi) = \{g | \varphi(g) = id_H\} = \{g | \ \forall h \in H, \ ghg^{-1} = h\} = C_G(H).$$

And our claim is proved.

6. Let us write the given equations in the form

$$ab^2a^{-1} = b^3 \qquad ba^2b^{-1} = a^3$$

Denote by $o(x)$ the order of x for every $x \in G$. From the equations above, we have $o(b^2) = o(b^3)$. It follows that

$$o(b^2) = o(b) = m.$$

The relation

$$o(x^k) = \frac{o(x)}{\gcd(k, o(x))}$$

implies that $\gcd(2, m) = 1$, and therefore there exists integers r, p such that $2r + mp = 1$. In a similar manner, we conclude that there exists integers s, t such that $2s + nt = 1$, where $n = o(a)$.

The condition $ab^2a^{-1} = b^3$ implies that $b^{3r} = ab^{2r}a^{-1} = aba^{-1}$. Similarly, $a^{3s} = bab^{-1}$. But $b^{3r} = b^{2r} \cdot b^r = b^{r+1}$, and the first condition becomes

$$b^r = aba^{-1}b^{-1} = a \cdot (bab^{-1})^{-1} = a \cdot a^{-3s} = a \cdot a^{-s-1} = a^{-s}.$$

It follows that $b = b^{2r} = a^{-2s} = a^{-1}$, and now it is easy to conclude that $a = b = 1$.

7. Let $G' =< a >$, that is, G' contains the unit element and a. We will start by showing that $a \in Z(G)$. Indeed, for every $x \in G$, $axa^{-1}x^{-1} \in G'$ which means that either $axa^{-1}x^{-1} = 1$ or $axa^{-1}x^{-1} = a$. The latter option implies that $a = 1$, which is impossible, and thus $ax = xa$ for every $x \in G$.

For every $g \in G$ denote by

$$G \cdot g = \{y | \exists x \in G, xgx^{-1} = y\}.$$

Suppose $y \in G \cdot g$. Then either $y = g$, or there exists $x \in G$ such that $xg \neq gx$, and $xgx^{-1} = y$. We conclude that

$$a = xgx^{-1}g^{-1} = yg^{-1}$$

and thus $y = ag$. Therefore, $G \cdot g = \{g\}$ if $g \in Z(G)$ and gG' otherwise.

Obviously, $Z(G) \neq G$ because otherwise G is Abelian, and $G' = \{1\}$. Let $g \notin Z(G)$. It is well known that $|G \cdot g| = |G|/|Z_g|$, the notation Z_g standing for $\{x | xg = gx\}$. Since $G \cdot g = gG'$, we end

up with $|G| = 2|Z_g|$. Suppose $x \in Z_g$. Then $xg = gx$, and since $a \in Z(G)$,

$$(ax)g = a(xg) = (xg)a = g(xa) = g(ax)$$

So that $ax \in Z_g$. We conclude that Z_g contains a natural number of cosets of G', and therefore $|Z_g|$ is even. Finally, $|G|$ is divisible by 4 and $|G : G'| = \frac{1}{2}|G|$ is even.

1994

1. We demonstrate an algorithm A which will be used for the solution:

(a) Find the median h of the set of all weights.

(b) Check, using the BFS algorithm if the graph

$$G_1 = (V, \{e \in E | w(e) \leq h\})$$

is connected.

(c) If the answer to 2 is NO, then let V' be the set of connection compounds of G_1, and E' be the set of edges connecting between those compounds, and return the result of A applied on the graph (V', E').

(d) If the answer to 2 is YES, then check if the graph

$$G_2 = (V, \{e \in E | w(e) < h\})$$

is connected using the BFS algorithm.

(e) If the answer to 4 is NO, then return h.

(f) If the answer to 4 is YES, then return the result of A applied on G_2.

The required algorithm is the following:

(a) Apply A on the graph, and let w_1 be the number it returns.

(b) Find a spanning tree to the graph

$$(V, \{e \in E | w(e) \leq w_1\}).$$

This tree is the required spanning tree of the graph.

Correctness of the algorithm: Each time A is applied, either it stops or it runs on a graph with number of edges at least twice as small. Therefore A must stop, and return a number w_1. We shall prove that this number is the minimal weight w for which the graph with edges whose weight is at most w is connected. It is easy to see that if this number is the median h, then the algorithm returns the correct answer. But since this number is not changed each time that we create a new graph in our algorithm, it follows that A indeed returns the required number. Hence follows the correctness of the whole algorithm.

Complexity: We shall prove by induction on m that the complexity of the algorithm is indeed $O(m)$. The case $m = 1$ is trivial. Suppose

that the complexity of the algorithm on a graph with less than m edges is $O(|E|)$. Let c be a constant such that the algorithm takes at most $c|E|$ steps, and we can assume that $c|E|$ is always greater than twice the number of steps it takes to do stages $1, 2, 3, 4, 5, 6$ in A, without recursive calls. This is possible, for all the stages in A take $O(|E|)$ steps. Then for $m \geq |E| < 2m$:

If the algorithm stops at stage 5 of the first call, then obviously we are through. Otherwise, in the next call the algorithm is applied on a graph with less than half the number of edges, and thus the total number of steps is at most

$$
\begin{aligned}
& \text{(The number of steps to do } 1, 2, 3, 4, 5, 6) + c\frac{|E|}{2} \\
< \quad & c\frac{|E|}{2} + c\frac{|E|}{2} \\
= \quad & c|E|.
\end{aligned}
$$

And we are done.

2. The algorithm:

(a) Turn G into $G' = (V, \{f \in E | w(f) < w(e)\})$.

(b) Check if there is a path in G' between x and y.

(c) If the answer to 2 is YES, then return NO.

(d) Else, return YES.

Correctness of the algorithm: We shall prove that there is a spanning tree in the graph containing e if and only if the algorithm returns YES.

If there is such a spanning tree, then it is impossible that there is a circuit in the graph in which e is the heaviest edge. For otherwise, remove e from the alleged spanning tree and add one of the edges in the circuit so that this tree will be connected. We get a lighter spanning tree, which contradicts the minimality of the spanning tree.

If the algorithm returns YES, then there is no spanning tree with all its weights smaller than $w(e)$. Apply the Kruncal algorithm to find a spanning tree, with e being the first edge in the list of edges with weight $w(e)$. Then we would get a minimal spanning tree, necessarily containing e.

Complexity: The complexity of the algorithm is clearly $O(|E|)$.

3. The algorithm:

(a) Check, using BFS algorithm, for which vertices x it is possible to reach x from s.

(b) Check that x cannot be reached from s if and only if $d(x) = \infty$.

(c) If 2 does not hold, return NO and stop.

(d) Check if $d(s) = 0$. If no, return NO and stop.

(e) For each $y \neq s$ with $d(y) \neq \infty$:

  i. Check for every vertex $z \xrightarrow{e} y$ if $d(z) + w(e) \geq d(y)$. If no, return NO and stop.

  ii. Check if there is a vertex $z \xrightarrow{e} y$ with $d(z) + w(e) = d(y)$. If no, return NO and stop.

(f) If you have reached so far, return YES and stop.

Correctness of the algorithm: Assume that d is indeed the function assigning the weight of the path from s to x for every vertex $x \in V$. It is very easy to see that all the tests in stages $2, 4, 5$ indeed hold for d and thus the algorithm will return YES.

Assume that the algorithm returns YES. The test in stage 2 implies that d is correct for all the vertices which cannot be reached from s. The test at stage 4 implies that d is correct to s.

Define a function $f : V \to R$ which gives for every vertex $x \in V$ the minimal weights of the paths from s to x. We need to show that $f(y) = d(y)$ for all y that can be reached from s.

First, we show that $f \geq d$. Assume by contradiction that this is not the case, and let y be the vertex for which $f(y) < d(y)$ and $f(y)$ is minimal. Then $y \neq s$. There is a path from s to y with weight $f(y)$. Let z be the preceding vertex to y in that path, and e be the edge (z, y). Then

$$f(z) + w(e) = f(y).$$

By the minimality of y, it follows that $f(z) = d(z)$. But then

$$d(y) > f(y) = d(z) + w(e),$$

which contradicts 5.1.

Next, we show that $f \leq d$. Assume by contradiction that there is a vertex y for which $f(y) < d(y)$. Let y be the vertex with the

minimal $d(y)$ that satisfy this. Then $y \neq s$. From 5.2, it follows that there is a vertex $z \xrightarrow{e} y$ such that

$$d(z) + w(e) = d(y).$$

Then z can be reached from s, and $d(z) < d(y)$. It follows that $d(z) = f(z)$, but then, from the definition of f,

$$f(y) \leq f(z) + w(e) = d(y),$$

a contradiction.

Therefore $f \leq d$ for all the vertices y.

Finally, we have proved that $f \equiv d$.

Complexity: Stages $1, 2$ can be done in $O(n)$ steps. Stages $3, 4, 6$ are of complexity $O(1)$. In stage 5, we check each edge in the graph at most twice, and hence it requires $O(m)$ steps. Therefore the complexity of the whole algorithm is $O(m + n)$.

4. (a) Add a vertex to G and connect it to all the other vertices. The new graph is connected and the degrees of all vertices are even (since the degree of every vertex in the old graph is 4, and the degree of the new vertex is equal to the number of the vertices in the old graph, which is even). Hence, it contains an Euler circuit.

Walk on this circuit and colour its edges alternately in two colours. Then remove the new vertex, and let E_1 be the set of the remaining edges coloured in the first colour, and E_2 the set of the remaining edges coloured in the second colour. It is clear that for each vertex, there are at most two edges in the same colour connecting it, and therefore we obtained the required result.

(b) It is clear that the last algorithm can be applied in $O(m)$ steps.

(c) First, colour the vertices of G in four colours $0, 1, 2, 3$ using the greedy algorithm – for each vertex, colour it such that its colour will be different from the colour of all its neighbors. This can clearly be done in $O(n)$ steps, where $n = |V|$, and since $3n = 2m$, $O(n) = O(m)$.

Next colour the edges of G the following way: let $c(v)$ denote the colour of a vertex $v \in V$. For each edge $e(x, y)$, colour e in $c(x) + c(y) \bmod 4$. One can easily show, that if

$e_1(x, y_1), e_2(x, y_2)$ are edges having a common vertex x, then since $c(y_1) \neq c(y_2)$, it follows that

$$c(x) + c(y_1) \not\equiv c(x) + c(y_2) \qquad \text{mod } 4.$$

Therefore our colouring is good.

5. *(This solution is due to Prof. Noga Alon of Tel-Aviv University)*

We shall prove by induction on k that if $n \geq 2^{k-1}$ then it requires at least k questions to determine the product of $x_1, \ldots x_n$.

For $k = 1$ the claim holds, for if $n \geq 1$ then it takes at least one question to determine the sign of x_1.

Assume that the induction claim holds for k, and let $n \geq 2^k$. Suppose that the first question is whether

$$\sum_{i=1}^{n} a_i x_i > c.$$

If

$$-\sum_{i=1}^{[n/2]} |a_i| + \sum_{i=[n/2]+1}^{n} |a_i| \geq c$$

then the answer to that question is positive for every sequence

$$(x_1, x_2, \ldots x_{[n/2]}, \text{sign } a_{[n/2]+1}, \ldots \text{sign } a_n).$$

Hence determining the product of the original sequence using linear questions is equivalent to finding that product for the sequence $(x_1, x_2, \ldots, x_{[n/2]})$. By our induction hypothesis, it will take at least k more questions, making it a total of $k + 1$ questions.

On the other hand, if

$$-\sum_{i=1}^{[n/2]} |a_i| + \sum_{i=[n/2]+1}^{n} |a_i| < c$$

then the answer to the question will be negative for every sequence

$$(-\text{sign } a_1, \ldots, -\text{sign } a_{[n/2]}, x_1, x_2, \ldots, x_{n-[n/2]})$$

and since $n - [n/2] \geq 2^{k-1}$, it follows from the induction hypothesis that we need at least k more questions.

1995

1. (a) We choose some $a_0 > 0$ and a real number c. Let P be the intersection point of the curve $x^4 - x^2y = a_0$ with the line $x = c$. Then the y-coordinate of P satisfies

$$y_c = c^2 - \frac{a_0}{c^2}.$$

The derivative of the curve is

$$4x^3 - 2xy - x^2y' = 0 \Rightarrow \; y' = 4x - 2\frac{y}{x}.$$

The tangent to the curve at P is therefore

$$\frac{y - y_c}{x - c} = 4c - 2\frac{y_c}{c}.$$

Simplifying, we get

$$y = \left(2c + \frac{2a_0}{c^3}\right)x - c^2 - \frac{3a_0}{c^2}.$$

At the point $x = \frac{3c}{2}$ the value of y is $y = 2c^2$, and therefore all of the tangent lines pass through the point $P_l = (\frac{3c}{2}, 2c^2)$.

(b) The coordinates of P_l are $P_l = (x_l, y_l) = (\frac{3c}{2}, 2c^2)$, and it follows that

$$y_l = 2c^2 = 2(\frac{2}{3}x)^2 = \frac{8}{3}x^2.$$

Thus the locus of the points P_l is the parabola $y = \frac{8}{3}x^2$.

2. Let $P = (x_0, y_0)$ with

$$x_0^{\frac{2}{3}} + y_0^{\frac{2}{3}} = 1.$$

Denote $u = \sqrt[3]{x_0}$, $v = \sqrt[3]{y_0}$, so that $u^2 + v^2 = 1$. We are now to find the tangent to the curve at P.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1 \Rightarrow \frac{2}{3}(\frac{1}{u} + \frac{1}{v} \cdot y') = 0 \Rightarrow y' = -\frac{v}{u}.$$

The tangent equation is therefore

$$uy + vx = uv.$$

The point Q of intersection between the tangent and the y-axis is $(0, v)$. To find R, we solve the two equations

$$\begin{cases} uy + vx = uv \\ x^2 + (y - v)^2 = b^2 \end{cases}.$$

The positive solution is $R = (|u|b, v - vb)$. The coordinates of R satisfy the equation

$$\frac{x^2}{b^2} + \frac{y^2}{(1-b)^2} = 1.$$

So R lies on the right half of this ellipse.

To end the proof, we need to show that for every point R' on the indicated ellipse, there exists a point P' on the curve such that $P'R'$ is tangent to the curve and the distance between R' to the intersection between $P'R'$ and the y-axis is b. Let $R' = (x_1, y_1)$. The corresponding point P' will be

$$P' = \left(\left(\frac{x_1}{b}\right)^3, \left(\frac{y_1}{1-b}\right)^3\right).$$

It is easy to verify that P' is indeed on the curve, and that $P'R'$ is tangent to the curve. The equation of $P'R'$ is

$$x_1(1-b)y + y_1 bx = x_1 y_1.$$

Then

$$Q' = \left(\frac{x_1}{b}, 0\right)$$

and

$$Q'R' = \sqrt{\left(x_1 - \frac{x_1}{b}\right)^2 + y_1^2} = b\sqrt{\frac{x_1^2}{b^2} + \frac{y_1^2}{(1-b)^2}} = b.$$

3. For every $P = (a, b, c)$, we will check if the projection of C from P on the x, y plane gives a curve with a cusp or a node.

A node will occur if one of the lines through P cuts two points on C, and hence there exist two points $(t, t^2, t^3), (s, s^2, s^3) \in C$ such that

$$\frac{t-s}{a-s} =^{(1)} \frac{t^2 - s^2}{b - s^2} =^{(2)} \frac{t^3 - s^3}{c - s^3}.$$

Equality (1) implies

$$b - s^2 = (t+s)(a-s)$$

and hence

$$t = \frac{b - as}{a - s}.$$

From equality (2) we deduce

$$s^2(a^2 - b) + s(c - ab) + (b^2 - ac) = 0.$$

The existence of a point P with a note is equivalent to the existence of a solution to the equation, i.e that the discriminant Δ of the equation is nonnegative.

$$\Delta = (c - ab)^2 - 4(a^2 - b)(b^2 - ac) \geq 0.$$

If $\Delta > 0$, then we have a node. If $\Delta = 0$, then we have an extremum node, which is a cusp.

1996

1. The maximal number of edges is 96. An example of such a graph is the bipartite graph 8×12. We shall prove that there can be no more then 96 edges in G.

Obviously, if the degree of each vertex of G is lesser than 12, then it can be at most 8, and the total number of edges is at most $\frac{1}{2} \times 8 \times 20 = 80$.

Suppose that there exists a vertex v such that the degree of v is greater than 12. Then the degree of v is 16, and let A be the set of vertices that are connected to v. No two vertices of A are connected, and thus the degree of each vertex in A is at most 4. The maximal degree of every vertex in G is at most 16, so that the degree of the three remaining vertices is lesser than or equal to 16. The total number of edges in G is therefore

$$\frac{1}{2}\sum_{x\in G} \deg(x) \le \frac{1}{2}(16 \times 4 + 4 \times 16) = 64.$$

Assume that the degree of each vertex in G is not greater than 12. Let v be a vertex with degree 12. Denote by A the set of those vertices of G which are connected to v and by B the set of the vertices outside of A. As before, no two vertices in A are connected, so their degree can be no more than 8. The degree of each vertex of G can be at most 12, and thus the degree of each vertex in B can be at most 12.

This brings the total number of edges in G to

$$\frac{1}{2}\sum_{x\in G} \deg x \le \frac{1}{2}(12 \times 8 + 8 \times 12) = 96.$$

The maximal number of edges in G can therefore be 96.

2. Consider the following algorithm to find such a subset S of G.

Set $i = 1$ and let $B_0 = G$, $n_0 = n$.

For every i, denote by B_i the subset of B_{i-1} of vertices not connected to v_i, and $n_i = |B_i|$.

We shall prove that there exists a sequence of vertices $v_1, v_2 \dots v_k$ such that $n_i \le \frac{n}{2^i}$.

Our set S will be the set of vertices $\{v_1, v_2, \dots v_k\}$ and since

$$k < 1 + \log_2 n,$$

this would yield the desired result.

For every i, consider two cases:

Case 1:

There exist a vertex x in B_{i-1} such that x is connected to at least $\frac{n_{i-1}-1}{2}$ vertices in B_{i-1}. Then set $v_i = x$, and then

$$n_i \leq \frac{n_{i-1}}{2},$$

thus

$$n_i \leq \frac{n}{2^i}.$$

Case 2:

If each vertex in B_{i-1} is connected to less then $\frac{n_{i-1}-1}{2}$ vertices in B_{i-1}, then each vertex is connected to at least $\frac{n-n_{i-1}+1}{2}$ vertices in A_{i-1}, the complementary set to B_{i-1}. The total number of edges in the bipartite graph between A_{i-1} and B_{i-1} is therefore $\frac{n_{i-1}(n-n_{i-1})}{2}$, and by the pigeonhole principle there exists a vertex $a \in A_{i-1}$ such that a is connected to at least

$$\frac{n_{i-1}(n - n_{i-1})}{2(n - n_{i-1} - 1)} \geq \frac{n_{i-1}}{2}.$$

Obviously, $a \neq v_j$ for every $j < i$, since every vertex in B_{i-1} is also in B_j. Set $v_i = a$, and we have

$$n_i \leq \frac{n_{i-1}}{2},$$

which implies

$$n_i \leq \frac{n}{2^i}.$$

Since the number of vertices in G is finite, the algorithm must end at some stage, say after k steps. Then $2^k < n$, or $k < 1 + \log_2 n$, and we have reached a set S as requested.

3. We shall prove that there is a Hamilton circuit in every connected k-regular bipartite graph if and only if $k = 2$.

For $k = 2$, every such graph is necessarily a circuit, and hence it contains a Hamilton circuit.

Assume that $k \neq 2$. Consider the following k-regular graph:

A complete k bipartite graph is a graph with $2k$ vertices, each party contains k vertices and each vertex is connected to all of the vertices in the other party. Take three such graphs K_1, K_2, K_3 together. Choose k pairs $(a_i, b_i)_{i=1}^k$ such that a_i is connected to b_i, $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$, and for each $j = 1, 2, 3$ there is at least one pair $(a_{i_j}, b_{i_j}) \in K_j$. Remove the edges (a_i, b_i), and let $\tilde{K_1}, \tilde{K_2}, \tilde{K_3}$ be the complete graphs after the removal of those edges. Now add two more vertices a, b. For $i = 1, 2, \ldots k$, connect a to b_i and b to a_i. Consider the graph G that was obtained. Then G is connected, and it is k-regular.

Assume by contradiction that there is a Hamilton circuit in G. Then this circuit passes through a and b. Remove a and b from G. The Hamilton circuit will be separated into two connecting compounds, but this is impossible since if we remove a, b from G we would get at least three connecting compounds $\tilde{K_1}, \tilde{K_2}, \tilde{K_3}$. This contradicts the assumption, and hence there is no Hamilton circuit in G.

4. Note that the given condition is equivalent to that from every subset T of vertices containing s but not t and there are at least k edges exiting T. Also note that if there are m edges entering some set X of vertices, then since the in-degree of each vertex is equal to its out-degree, there are m edges leaving X. s and t are therefore symmetric, so we can prove only one direction, that there are k path-disjoint paths from s to t.

We shall prove the claim using induction on n, the number of vertices in the graph. Obviously, $n \geq k+1$. The case $n = k+1$ is trivial, and assume that the claim holds for every graph with $m < n$ vertices. There are at least k edges coming from s, to at least k different vertices. Denote by T the subset of those vertices and s. We will now prove that there are at least k path-disjoint paths from s leaving T. Assume by contradiction that this is not the case. There are at least k edges leaving T, so at least two of them must leave the same vertex $a \in T$, with $a \neq s$. Consider the set of vertices $T \setminus \{a\}$. There are at most $k-2$ edges leaving T plus one more leaving s towards a, making it at most $k-1$ edges leaving $T \setminus \{a\}$, a contradiction.

Let us consider now the graph consisting of all the vertices of the original graph except for those in T, and T as a vertex. In this new graph, the in-degree of each vertex is equal to its out-degree, and for every set of vertices containing T there are at least k different edges

leaving it. By induction, there are k path-disjoint paths leaving T to t, and since the k paths leaving T are also path-disjoint, this means that the claim holds in the original graph. Therefore, the claim holds for every number n of vertices in the graph.

5. We will prove by induction that all those graphs with n vertices that do not contain a path with length 100 have no more than $49.5n$ edges, and that equality happens only in the graph consisting of unconnected K_{100} graphs.

The claim obviously holds for $n \leq 100$. Assume that it holds for all $k < n$, and let G be a graph with n vertices that do not contain a path with length 100. If there is a vertex in G whose degree is not greater than 49, then remove this vertex and proceed by applying the induction assumption on the remaining graph. Therefore, suppose that the degree of every vertex in G is greater than 49.

Consider the maximal path $x_1, x_2, \ldots x_m$ in G, with $m \leq 100$. All the neighbors of x_1 and x_m are among those vertices in the path, because otherwise we would get a contradiction to the maximality of m. Since x_1, x_m has more than 50 neighbors, it follows from the pigeonhole principle that there exist $1 \leq i < m$ so that x_i, x_m are neighbors and x_1, x_{i+1} are neighbors. Then

$$x_1, x_2, \ldots x_i, x_m, x_{m-1}, \ldots x_{i+1}, x_1$$

is a circuit. If there exists a vertex y connected to one of the vertices in the circuit, then there is a path with length $m + 1$, contradicting the maximality of m. Therefore in this connection compound the only vertices are the ones in the circuit. Therefore in this compound there is at most

$$\binom{m}{2} \leq 49.5m$$

edges. For the rest of the graph we apply the induction hypothesis, and find that it has at most $49.5(n - m)$ edges. Hence G has at most $49.5n$ edges.

Equality happens only if $m = 100$ and this connection compound is a complete graph. By induction, the rest of the graph must also consist of K_{100} connecting compounds.

In particular, if $n = 1000$ then the graph with he most edges consist of 10 unconnected K_{100} graphs.

6. We start by finding a lower bound to $f(k)$. To that end, we have to find an example for a connected graph with k vertices that does not contain a path with length 100. Consider the regular bipartite graph with 49 vertices on one party and $k-49$ vertices on the other. It is clear that this graph is connected, and since each path of length m contains at least $[m/2]$ vertices from the first party, there is no path of length 100 in the graph. In this graph there are

$$49(k-49) = 49k - 2401$$

edges.

We now note that we can add to this graph all of the edges connecting each two vertices from the first party, and the longest path in the graph will still be less than 100 in length. The number of edges in this graph is

$$49(k-49) + \binom{49}{2} = 49k - 1225$$

and hence $c_1 = 1225$.

We now turn to the task of finding an upper bound to $f(k)$.

Definition: *A graph G is said to be k-connected if for every $k-1$ vertices in G, the graph received by removing those vertices is connected, or equivalently, G cannot be partitioned into 2 subgraphs G_1, G_2 such that*

$$|V(G_1) \cap V(G_2)| \leq k-1$$

and if v is a vertex not in $V(G_1) \cap V(G_2)$ then v is not connected to the vertices in the other subgraph.

Lemma 1: *G is a simple, 2-connected graph with n vertices. If the degree of each vertex is at least k, then G contains either an Hamiltonian circuit or a circuit with at least $2k$ vertices.*

Proof: Let $P = (x_0, x_1, \ldots x_m)$ be the longest path in the graph. Clearly, all of the neighbors of x_0, x_m are in P. Consider three cases:

Case I: There does not exist $1 \leq i < j \leq m-1$ such that x_j is connected to x_0, x_i is connected to x_m. Let i be the maximal index such that x_i is connected to x_0, and let j be the minimal index for which x_j is connected to x_m. Then $j > i$ and all vertices $x_{i+1}, x_{i+2} \ldots x_{j-1}$ are not connected to either x_m or x_0.

Consider the sets $A_1 = \{x_1, x_2, \dots x_i\}, A_2 = \{x_j, x_{j+1}, \dots x_{m-1}\}$. The neighbors of x_0 are all A_1 while all the neighbors of x_m are in A_2, and since the degree of each vertex in G is at least k, it follows that $|A_1|, |A_2| \geq k$. Now in a 2-connected graph, every two vertices are connected by some simple circuit, and therefore there exist two disjoint paths P_1, P_2 connecting A_1 and A_2, say $P_1 = (x_p, \dots x_q), P_2 = (x_t, \dots x_s)$ with $p, t \leq i$ and $s, q \geq j$, with x_p, x_t connected to x_0 and x_t, x_s connected to x_m. Then we can find a circuit passing through the neighbors of x_0, x_m and through P_1, P_2, which implies that its size is greater than $2k$.

Case 2: There exists an index $0 \leq i \leq m-1$ such that x_i is connected to x_m and x_{i+1} is connected to x_0. Assume that there is a vertex v in G such that $v \notin P$. Then since G is connected, v is connected to x_0 and hence is connected to some vertex x_j in P. Without loss of generality, we can assume $j \leq i$. But then

$$(v, \dots, x_j, x_{j+1}, \dots x_i, x_m, x_{m-1}, \dots x_{i+1}, x_0, x_1 \dots, x_{j-1})$$

is a path with length $m+1$, contradicting the maximality of P. It follows that the circuit

$$(x_0, x_1, \dots x_i, x_m, x_{m-1}, \dots x_{i+1}, x_0)$$

is Hamiltonian.

Case 3: There exist two vertices x_i, x_j in P such that $i < j$, x_i is connected to x_m, x_j is connected to x_0 and the difference $j - i \geq 2$ is minimal. Then the vertices $x_{i+1}, \dots x_{j-1}$ are not connected to either x_0 or x_m, because of the minimality of $j-i$. Since the degree of x_0, x_m is at least k, it follows that $i, m-j \geq k-1$ and thus the circuit

$$(x_0, x_1, \dots, x_i, x_m, x_{m-1}, \dots x_j, x_0)$$

is with length greater than or equal to $2k$.

Lemma 2: *If G is a graph with n vertices and over $\frac{k(n-1)}{2}$ edges, then G has a circuit with size at least $k+1$.*

Proof: Clearly, we must have $k > n$. The claim will be proved by induction on n. The claim holds for $n = k+1$, because a graph with $k+1$ vertices and $\frac{k^2}{2}$ edges, contains a Hamiltonian circuit.

Assume that the claim holds for all $m < n$, and let G be a graph with n vertices and over $\frac{k(n-1)}{2}$ edges. If there is a vertex in G with degree not greater than $\frac{k}{2}$, remove it and we get a graph with

$n-1$ vertices and more than $\frac{k(n-2)}{2}$ edges, and by our induction assumption, G has a circuit with size $k+1$.

Suppose that all the degrees in G are at least $\frac{k+1}{2}$. If G is not 2-connected, it follows that we can divide G into two graphs G_1, G_2 such that

$$|V(G_1) \cap V(G_2)| \leq 1$$

in the way described above. By the pigeonhole principle, one of G_1, G_2 must satisfy the condition of the lemma, or otherwise we would have

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| \\ &\leq \frac{k}{2}\left(|V(G_1)| + |V(G_2)|\right) \\ &\leq \frac{k(n-1)}{2}, \end{aligned}$$

a contradiction.

We can now turn to finding an upper bound for $f(k)$. We claim that

$$f(k) \leq 49k - 49.$$

Proof: The proof follows easily from the last lemma. If G has more than $49k-49$ edges, then from the previous lemma, it has a circuit of length 100. Since G is connected, it follows that this circuit is connected to at least one more vertex, and therefore it contains a path of length 100.

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1. (a) We use the following lemma: if M is a point on the side XY of triangle XYZ, then

$$\frac{XM}{YM} = \frac{XZ \sin \angle XZM}{YZ \sin \angle YZM}$$

For if h is the altitude from Z to XY, then

$$\frac{area(\triangle XZM)}{area(\triangle YZM)} = \frac{h \cdot XM}{h \cdot YM} = \frac{XZ \cdot ZM \cdot \sin \angle XZM}{YZ \cdot ZM \cdot \sin \angle YZM}.$$

Let AP_1, BP_1, CP_1 meet the opposite sides at D_1, E_1, F_1, respectively, and let their reflections meet the opposite sides at D_2, E_2, F_2, respectively. Then by the lemma,

$$\begin{aligned} & \frac{BD_2}{CD_2} \cdot \frac{CE_2}{AE_2} \cdot \frac{AF_2}{BF_2} \\ = & \frac{\sin \angle BAD_2}{\sin \angle CAD_2} \cdot \frac{\sin \angle CBE_2}{\sin \angle ABE_2} \cdot \frac{\sin \angle ACF_2}{\sin \angle BCF_2} \\ = & \frac{\sin \angle CAD_1}{\sin \angle BAD_1} \cdot \frac{\sin \angle ABE_1}{\sin \angle CBE_1} \cdot \frac{\sin \angle BCF_1}{\sin \angle ACF_1} \\ = & \frac{CD_1}{BD_1} \cdot \frac{AE_1}{CE_1} \cdot \frac{BF_1}{AF_1}, \end{aligned}$$

and the result follows immediately from the Ceva's theorem.

(b) The quadrilaterals $P_1A_1CB_1$ and $P_2A_2CB_2$ are cyclic, and hence

$$\begin{aligned} \angle A_1B_1C &= \angle A_1P_1C \\ &= 90 - \angle P_1CA_1 \\ &= 90 - \angle P_2CB_1 \\ &= \angle B_2P_2C \\ &= \angle B_2A_2C. \end{aligned}$$

Thus the quadrilateral $A_1A_2B_2B_1$ is cyclic. $P_1B_1B_2P_2$ is a right-angle trapezoid, and hence the perpendicular bisector of B_1B_2 meets P_1P_2 at M, the midpoint of P_1P_2. Similarly, the

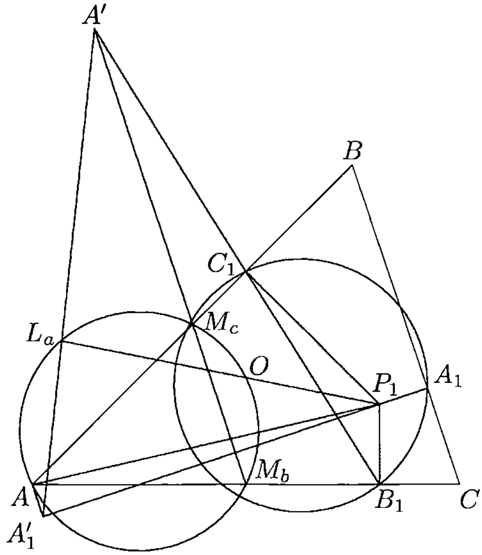
perpendicular bisector of A_1A_2 passes through M, and therefore M is the centre of the circle circumscribing $A_1A_2B_2B_1$. Similarly, the quadrilaterals $B_1B_2C_2C_1$ and $C_1C_2A_1A_2$ are cyclic, and the centre of the circumscribing circles is M. Hence $A_1A_2B_2B_1C_1C_2$ is inscribed in a circle with centre M.

(c) We shall prove the following claim, from which the problem is easily solved:

If P_1 lie on a fixed line passing through the circumcentre of the $\triangle ABC$, then the circle $A_1B_1C_1$ passes through a fixed point on the nine points circle of $\triangle ABC$.

Proof: Let M_a, M_b and M_c be the midpoints of the sides BC, CA and AB respectively. Denote by C', A' and B' the intersection points of M_aM_b, M_bM_c and M_cM_a with A_1B_1, B_1C_1 and C_1A_1 respectively. We intend to prove that the three lines A_1A', B_1B' and C_1C' meet at a point L which lies on the intersection of the nine points circle and of the circle $A_1B_1C_1$.

Consider the circumcircle AM_bM_c. Let L_a be its second intersection point with the line P_1O (O being the circumcentre of $\triangle ABC$). Then $\angle AL_aO = 90°$. Hence the points L_a, B_1, C_1 lie on the circle with AP_1 as diameter. Let A_1' be the reflection of A_1 with respect to M_bM_c. Then A_1' lies on the line passing through A and parallel to BC, and also on the line A_1P_1. Hence $\angle AA_1'P_1 = 90°$, so that A_1' also lies on the circle with AP_1 as diameter.



We shall now prove that A', L_a, A_1' are collinear. Since L_a lies on the circles AM_bM_c and AB_1C_1, it follows from the theorem of Simson that the feet of the perpendiculars from L_a to AM_b, AM_c, A_1B_1 and M_aM_b are collinear. Hence L_a lies on the circumcircle of $A'B_1M_b$ by the converse of this theorem. Thus the angles $\angle A'L_aB_1, \angle A'M_bB_1$ are either equal or their sum is equal to $180°$. We shall consider the first case, and the proof to the latter case is similar, with the appropriate modifications.

Since A, P, B_1 , C_1, L_a and A_1' are concyclic, it follows that

$$\begin{aligned} \angle B_1L_aA_1' &= \angle B_1AA_1' \\ &= 180° - \angle A_1'PB_1 \\ &= 180° - \angle A'M_bB_1. \end{aligned}$$

Furthermore,

$$\angle A'L_aB_1 = 180° - \angle B_1L_aA_1'.$$

Therefore A', L_a and A_1' are collinear. Let L be the reflection of L_a with respect of M_bM_c. Then L lies on A_1A' and also on the nine points circle (since reflection through M_bM_c maps the circle AM_bM_c to the nine points circle). Also,

$$\overrightarrow{AL} \cdot \overrightarrow{A'A_1} = \overrightarrow{A'L_a} \cdot \overrightarrow{AA_1'} = \overrightarrow{AC_1} \cdot \overrightarrow{A'B_1}.$$

Hence L is the intersection of the nine points circle and the circle $A_1B_1C_1$. Thus if P_1 moves along a fixed line through O, then the point L_a, and therefore L, remains fixed.

2. First Solution:

Assume that the ant moves in a direction parallel to the x-axis, and starts from the point (a, b) on the given curve. It meets the curve at the (c, b), where we have

$$\begin{cases} a^2 + ab + b^2 = 6 \\ c^2 + cb + b^2 = 6 \end{cases}$$

Subtracting the first equation from the second, we get

$$(a - c)(a + b + c) = 0,$$

which implies $c = -a - b$. Similarly, if the ant moves in a direction parallel to the y-axis, it meets the curve at $(a, -a - b)$.

Let $P = (a, b)$ be the starting point of the ant, and assume that the ant starts walking in a direction parallel to the x-axis (the second case is analogous). If at some stage during the walk, the ant cannot move any longer, it stops. Otherwise, its walk is composed of the following points on the curve:

$$\begin{aligned} (a, b) \quad \rightarrow \quad & (-a - b, b) \rightarrow (-a - b, a) \rightarrow (b, a) \rightarrow (b, -a - b) \\ & \rightarrow (a, -a - b) \rightarrow (a, b). \end{aligned}$$

Therefore, the ant returns to the starting point of the walk after at most six steps.

Second Solution:

Rotate the curve by 45°, by multiplying (x, y) with the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The equation of the image of the curve under the rotation is

$$3x^2 + y^2 = 12.$$

Hence the curve is an ellipse. The new directions of the paths of the ant are inclined by $\pm 45°$ with respect to the x-axis.

Apply an affine transformation, sending y into $\frac{1}{\sqrt{3}}y$. The curve is mapped to the curve

$$x^2 + y^2 = 4$$

which is a circle with radius 2. The directions of the paths of the ant are now inclined by $\pm 30^\circ$ with respect to the x-axis.

Denote the centre of the circle by O. Assume that the ant starts moving while creating an angle of 30° with respect to the x-axis from P to P_1 and from P_1 to P_2. Then

$$\angle PP_1P_2 = 120^\circ \Rightarrow \angle POP_2 = 120^\circ$$

Hence, moving twice, the ant rotates around O by 120°. Therfore, after at most six moves, the ant will return to the starting point.

3. (a) The answer is NO. Assume by contradiction that this construction can be made using a ruler only. Then the construction is made using a sequence of straight lines and their point of intersection.

Perform a projective transformation on the figure, taking C to another circle C'. The lines and their points of intersection are kept under the transformation, and thus they yield the point which is the map of the centre of C. On the other hand, those lines compose a construction which gives the centre of C'. But the centre of C is not mapped to the centre of C', and thus we arrive at a contradiction.

(b) i. Denote by P and Q the intersection points of C_1 and C_2. We shall describe a construction which leads to finding both centres O_1 and O_2 of C_1, C_2, and all that remains is to construct O_1O_2.

Choose a point A inside C_1. Construct AP, AQ and let B, C be their second intersection with C_1, and D, E their second intersection with C_2. Then

$$\angle ADE = \angle AQP = \angle ACB$$

Thus, $DE \| BC$. Construct DQ, EP and let F, G be their second intersection with C_1. Then,

$$\angle GFQ = \angle EPQ = \angle EDQ$$

Hence, $DE \| FG$, and it follows that $FG \| BC$.

Since B, C, F and G lie on C_1, then $BCGF$ is an isosceles trapezoid. Construct H and I, which are the point of intersection of BF and CG, and of BG and CF, respectively. Then $HB = HC$, $IB = IC$ which implies that HI is the perpendicular bisector of BC, so it passes through O_1.

Choose another point A' inside C_1. The same construction will yield another line through O_1, and then we can construct O_1. Similarly, we can construct O_2, and then we can construct O_1O_2.

ii. Construct two lines through T. Those lines meet C_1 at A, B and C_2 at C, D. We shall prove that $AB||CD$. Let P, Q be points on the common tangent from different sides of T. Then

$$\angle ABT = \angle ATQ = \angle CTP = \angle CDT$$

Hence $AB||BC$. Construct AD, BC and their point of intersection S. Then S is the external centre of similarity of C_1 and C_2, and hence ST is the line of the centres of C_1, C_2.

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1. We shall use induction on n. The case $n = 3$ is trivial. Assume that the claim holds for $n-1$, and consider K_n. Let a be a vertex in K_n, and consider

$$K_n \setminus \{a\} \cong K_{n-1}.$$

If it is coloured with at least $n-1$ colours, then by the induction hypothesis we are done. Otherwise, it is coloured with at most $n-2$ colours, say $1, 2, \ldots n-2$. The colours $n-1, n$ must appear in K_n and hence there exist two vertices $b, c \neq a$ such that the edge (a, b) is coloured with colour $n-1$ and the edge (a, c) is coloured with n.

But (b, c) is not coloured with either $n-1$ or n and therefore a, b, c is a triangle whose sides are coloured with three different colours.

2. We shall use induction on n. For $n = 5$,

$$e(G_5) \geq \frac{25}{4} + 2$$

implies

$$e(G_n) \geq 9,$$

so at most one pair of vertices in G_5 is not connected. Let a, b, c, d, e be the vertices and suppose that a, b are not connected. Then $\{a, e, c\}$ and $\{b, d, c\}$ are two triangles with one common vertex.

Assume that the claim holds for $n-1$, and consider G_n with

$$e(G_n) \geq \frac{n^2}{4} + 2.$$

If the degree of some vertex in G_n is lesser than $\frac{n}{2}$, then we can remove it and the number of edges in the new graph is at least

$$\frac{n^2}{4} + 2 - \frac{n}{2} + 1 \geq \frac{(n-1)^2}{4} + 2,$$

and we can apply the induction hypothesis. Therefore we can assume that the degree of every vertex is at least $\frac{n}{2}$.

Since the sum of the degrees in G_n is

$$\frac{n^2}{2} + 8 > \frac{n^2}{2}$$

then there exists at least one vertex a who has degree greater than $\frac{n}{2}$.

Then

$$\deg(a) \geq \lfloor \frac{n}{2} \rfloor + 1.$$

Let $v_1, \ldots v_r$ be the vertices connected to a, with

$$r \geq \lfloor \frac{n}{2} \rfloor + 1$$

and let $u_1, \ldots, u_s$ be the remaining vertices, with

$$s = n - r - 1 \leq \lceil \frac{n}{2} \rceil - 2.$$

Now $n \geq 6$ and hence $r \geq 4$. For every $1 \leq i \leq r$, the degree of v_i is at least $\frac{n}{2}$ and hence it is connected to at least one of the other vertex v_j.

Suppose that v_1 is connected to v_2. If v_3 is connected to a vertex v_j with $j \neq 1, 2$ then the triangles $\{a, v_1, v_2\}$ and $\{a, v_3, v_j\}$ have one common vertex. Hence we can assume v_3 is connected to v_1. For $4 \leq i \leq r$, if v_i is connected to a vertex v_j with $j \neq 1$ we are finished, and hence we can assume that v_1 is connected to all of the vertices $v_2, v_3, \ldots v_r$ and that no other pair v_i, v_j is connected. It follows that each vertex v_i for $i \neq 2$ is connected to all of the vertices u_j, $1 \leq j \leq s$ and that

$$s = \lceil \frac{n}{2} \rceil - 2.$$

Consider the sets $A = \{v_2, v_3 \ldots v_r\}$ and $B = \{a, v_1, u_1 \ldots u_s\}$, with $|A| = \lfloor \frac{n}{2} \rfloor$ and $|B| = \lceil \frac{n}{2} \rceil$. For every $x \in A, y \in B$ the pair x, y is connected. In addition, a, v_1 are connected and no pair from A is connected. Since

$$e(G_n) \geq \frac{n^2}{4} + 2$$

then at least one more pair in B must be connected, and therefore there exist two triangles with exactly one common vertex.

3. Assume by contradiction that the claim does not hold, so G_n does not contain C_4. Let $e = e(G_n)$ and denote by $d_1, d_2 \ldots d_n$ be the degrees of the vertices in G_n. Then

$$\sum_{i=1}^{n} d_n = 2e.$$

Let m be the number of triples (a, b, c) of vertices in G_n with a connected to both b, c (where (a, b, c) and (a, c, b) are considered identical). Since for every pair b, c there can be at most one vertex a that is connected to both b, c (for otherwise G_n contains C_4) and thus $m \leq \binom{n}{2}$. On the other hand, every vertex a is a member of $\binom{\deg(a)}{2}$ triples (a, b, c), and hence by the quadratic-arithmetic means inequality,

$$\frac{n(n-1)}{2} \geq m = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} = -e + \frac{1}{2}\sum_{i=1}^{n} d_i^2 \geq -e + \frac{2}{n}e^2.$$

Rearranging, we obtain $4e^2 - 2ne - n^2(n-1) \leq 0$. Hence

$$e \leq \frac{2n + \sqrt{4n^2 + 16n^2(n-1)}}{8} < \frac{n}{4} + \frac{n\sqrt{n}}{2}.$$

A contradiction.

4. (a) We shall obtain a stronger result, namely that

$$e(G_n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + \frac{n}{4}.$$

Let $d_1, d_2, \ldots d_n$, m and e be as in the previous problem. In this case, $m \leq 2\binom{n}{2}$ since every pair b, c can have at most two common neighbours. Then

$$n(n-1) \geq m = \sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} = -e + \frac{1}{2}\sum_{i=1}^{n} d_i^2 \geq -e + \frac{2}{n}e^2$$

and hence

$$2e^2 - ne + n^2(n-1) \leq 0.$$

Therefore

$$e \leq \frac{n + \sqrt{n^2 + 8n^2(n-1)}}{4} < \frac{n}{4} + \frac{n\sqrt{n}}{\sqrt{2}}.$$

(b) Here also we shall prove a stronger result: the claim holds for every value of n.

Consider the graph G_n which has the points $P_1, P_2, \ldots P_n$ as vertices and every two points P_i, P_j are connected if and only if P_iP_j has unit length.

If P, Q are two points in the plain then there are exactly two points in the plane that have unit distance from both P and

Q. Therefore, the graph G_n does not contain $K_{3,2}$ and hence by (a),

$$e(G_n) \le \frac{n}{4} + \frac{n\sqrt{n}}{\sqrt{2}} \le n\sqrt{n}$$

The last inequality holds for every $n \ge 2$. If $n = 1$ there is nothing to prove.

Comment: In fact, there is a stronger bound for $e(G_n)$, which is in $O(n\sqrt[3]{n})$.

5. In the following problem, equality means equality in the field of residues modulo the prime p.

(a) Assume by contradiction that (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4) form a C_4. Then

$$\begin{aligned} x_1x_2 + y_1y_2 &= 1 \\ x_2x_3 + y_2y_3 &= 1 \\ x_3x_4 + y_3y_4 &= 1 \\ x_4x_1 + y_4y_1 &= 1. \end{aligned}$$

Subtracting the first two equations and the last two equations yields

$$\begin{aligned} x_2(x_1 - x_3) &= -y_2(y_1 - y_3) \\ y_4(y_1 - y_3) &= -x_4(x_1 - x_3). \end{aligned}$$

Multiplying both equations, we obtain

$$(x_1 - x_3)(y_1 - y_3)(x_2y_4 - y_2x_4) = 0.$$

Similarly,

$$(x_2 - x_4)(y_2 - y_4)(x_1y_3 - y_1x_3) = 0.$$

If, for example $x_1 = x_3$ then $y_1y_2 = 1 - x_1x_2 = 1 - x_3x_2 = y_2y_3$ and hence either $y_1 = y_3$, which is impossible since $(x_1, y_1) \neq (x_3, y_3)$ by assumption, or $y_2 = 0$. Similarly, $y_4 = 0$. But then $x_2x_1 = 1 = x_1x_4$ and thus $x_2 = x_4$, a contradiction. We conclude that no two elements from the pairs $\{x_1, x_3\}$, $\{x_2, x_4\}$, $\{y_1, y_3\}$ and $\{y_2, y_4\}$ are equal.

Therefore $x_1y_3 = x_3y_1$ and $x_2y_4 = x_4y_2$. Using the first two equations, we obtain

$$x_3 = x_1x_2x_3 + y_1y_2x_3 = x_1x_2x_3 + x_1y_2y_3 = x_1.$$

And thus we arrive at a contradiction.

(b) We shall prove a stronger result, that there exist infinitely many values of n such that there exist a graph G_n that does not contain C_4 and

$$e(G_n) \geq \frac{n\sqrt{n}}{2} + \frac{5\sqrt{n}}{2}.$$

Consider the graph from part (a). The number of vertices in this graph is clearly $n = p^2$.

We shall investigate the degree of each vertex. Let $v = (x, y)$ be a vertex in G. If $x = 0, y = 0$ then v has degree 0. Otherwise, if $x \neq 0$ then for every y' there exists a unique

$$x' = x^{-1}(1 - yy')$$

such that (x', y') is connected to v. The case where $y \neq 0$ is similar, and in both cases the degree of v is p. Hence, the total number of edges in the graph plus the number of loops in the graph is equal to $\frac{1}{2}p(p^2 - 1)$. The number of loops in the graph can be at most $2p$, since for every x, there exists at most two values of y such that $1 - x^2 = y^2$. Hence G has at most

$$\frac{1}{2}p^3 - \frac{5}{2}p = \frac{1}{2}n\sqrt{n} - \frac{5}{2}\sqrt{n}.$$

7. CLASSIFICATION OF PROBLEMS BY TOPIC

Individual Problems

Plane Geometry

90/2, 91/2, 93/3, 94/3, 95/2, 97/3, 97/5, 98/5, 00/3, 01/2, 01/5.

Functions

91/1, 93/2, 95/3, 99/1, 99/3, 99/5, 01/3, 01/4.

Sequences

90/3, 91/3, 92/3, 93/2.

Trigonometry

98/2.

Number Theory

90/1, 93/1, 95/1, 96/1, 96/2, 97/1, 98/3, 98/4, 99/4, 00/2, 00/5, 01/1.

Equations and Inequalities

91/4, 92/1, 94/2, 96/4, 97/2, 00/6.

Combinatorics

92/2, 94/4, 95/4, 96/3, 97/4, 97/6, 98/1, 98/6, 99/2, 99/6, 00/1, 00/4.

Miscellaneous

90/4, 92/4, 93/4, 94/1, 01/6.

Team Problems

The topics of the team competition problems

1990 Continued fractions.

1991 Polynomials.

1992 Sequences.

1993 Groups.

1994 Algorithms in graph theory.

1995 Algebraic curves.

1996 Graph theory.

2000 Geometry (transformations, constructions).

2001 Extremal graph theory.

8. LIST OF CONTESTANTS

1990

Hungarian Contestants	Israeli Contestants
Balogh József	Alon Gil
Matolcsi Máté	Lapid Erez
Kondacs Attila	Ladkany Sephi
Czirók András	Meiraz Guy

1991

Hungarian Contestants	Israeli Contestants
Boncz András	Alon Gil
Harcos Gergely	Braverman Alexander
Turányi Zoltán	Gemintern Alexander
Ujváry- Menyhárt Zoltán	Ladkany Sephi

1992

Hungarian Contestants	Israeli Contestants
Ujváry-Menyhárt Zoltán	Angel Omer
Álmos Attila	Mangovy Dan
Futó Gábor	Pinhasy Rom
Szendröi Balázs	Vanunu Avishay

1993

Hungarian Contestants	Israeli Contestants
Katz Sándor	Angel Omer
Futó Gábor	Borde Yuri
Csörnyei Marianna	Nehushtan Oren
Kálmán Tamás	Vanunu Avishay

1994

Hungarian Contestants	Israeli Contestants
Csörnyei Marianna	Iorsh Maxim
Futó Gábor	Shkolnikov Hagay
Koblinger Egmont	Yager David
Párniczky Benedek	Zilberman Lior

1995

Hungarian Contestants	Israeli Contestants
Szádeczky-Kardoss Szabolcs	Iorsh Maxim
Koblinger Egmont	Buhovsky Lev
Valkó Benedek	Radzivilovsky Lev
Burcsi Péter	Zuk Or

1996

Hungarian Contestants	Israeli Contestants
Burcsi Péter	Buhovsky Lev
Gyarmati Katalin	Carmiel Yishay
Ba'ra'sz Miha'ly	Desiatnikov Eli
Gröller kos	Radzivilovsky Lev

1997

Hungarian Contestants	Israeli Contestants
Braun Gábor	Carmiel Yishay
Pap Gyula	Dovgard Roman
Frenkel Péter	Heller Yuval
Lippner Gábor	Yuval Tom
Tóth dm	

1998

Hungarian Contestants	Israeli Contestants
Lippner Gábor	Dovgard Roman
Lukács László	Keller Natan
Zubcsek Péter-Pál	Puder Doron
Bérczi Gergely	Simkin Michael

1999

Hungarian Contestants	Israeli Contestants
Gyenes Zoltán	Braverman Mark
Kiss Gergely	Lang Oran
Terpai Tamás	Tessler Ran
Zábrádi Gergely	Yuval Amitay

2000

Hungarian Contestants	Israeli Contestants
Csikvári Péter	Antin Alexey
Gyenes Zoltán	Braverman Mark
Vizer Máté	Lang Oran
Zábrádi Gergely	Tessler Ran

2001

Hungarian Contestants	Israeli Contestants
Csikvári Péter	Antin Alexey
Csóka Endre	Assaf Eran
Harangi Viktor	Lang Oran
Horváth Illés	Shafrir Doron
Vörös László	Shein Aviv
Vizer Tibor	Tessler Ran

9. GLOSSARY

In this chapter we cite several well known facts that were used in the book, and which may be nonstandard in regular high school curriculum.

The triangle inequality

For any two real numbers a, b,

$$|a+b| \leq |a| + |b|.$$

This inequality holds also for complex numbers. In such cases, when $z = x + iy$, $|z|$ can be interpreted as the distance of the point (x, y) from the origin.

The Cauchy-Schwartz inequality

For any two sequences of real numbers

$$a_1, a_2, \ldots, a_n \quad \text{and} \quad b_1, b_2, \ldots, b_n$$

of n the following inequality holds

$$\sum_{i=1}^{n} a_i b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2}.$$

Equality holds if there exists a real number λ such that $a_i = \lambda b_i$ for all $1 \leq i \leq n$.

Pell's equation

The diophantine equation

$$x^2 - dy^2 = 1$$

where the given integer d is not a perfect square.

If (x_0, y_0) is the smallest solution where x_0, and y_0 are positive, then all solutions can be expressed by

$$x + y\sqrt{d} = \pm(x_0 + y_0\sqrt{d})^n, \quad n \in \mathbb{N}.$$

Fibonacci sequence

The sequence F_n defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with $F_0 = F_1 = 1$ is called Fibonacci sequence.

F_n can be expressed explicitly by means of (Binet's formula)

$$F_n = \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Pythagorean triples

Three positive integers x, y, z satisfying $x^2 + y^2 = z^2$ are called a Pythagorean triple. The name stems from Pythagoras' Theorem satisfied by the sides x, y, z of a right angled triangle.

Any Pythagorean triple x, y, z for which x, y, z are relatively prime, can be expressed by $x = m^2 - n^2$, $y = 2mn$ (or $x = 2mn$, $y = m^2 - n^2$) and $z = m^2 + n^2$, for some relatively prime positive integers m, n.

The Sine Law

If $\triangle ABC$ is a triangle whose sides are a, b, c and the corresponding angles are α, β, γ, then

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R,$$

where R is the circumradius of $\triangle ABC$.

The Cosine Law

If $\triangle ABC$ is a triangle whose sides are a, b, c and the corresponding angles are α, β, γ, then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha, \\ b^2 &= c^2 + a^2 - 2ca\cos\beta, \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma. \end{aligned}$$

The inradius of a right angled triangle

Let a, b be the sides of a right angled triangle, c its hypotenuse and r its inradius. Then

$$r = \frac{1}{2}(a + b - c).$$

The relation between the inradius and the circumradius in a triangle

Let R and r be respectively, the circumradius and the inradius of $\triangle ABC$. Then $R \geq 2r$ and equality holds only for an equilateral triangle.

The product law for chords in a circle

Let PQ and RS be two chords in a circle C, and let T be their intersection point. Then

$$PT \cdot QT = RT \cdot ST = k.$$

If T is outside the circle C, and TU touches C at the point U, then $k = TU^2$.

The area of a triangle

Let ABC be a triangle with

$$\begin{array}{llll} \text{sides } a = BC, & b = CA, & c = AB, & \\ \text{angles } \alpha = \angle BAC, & \beta = \angle ABC, & \gamma = \angle CAB. \end{array}$$

Let the inradius and the circumradius be r and R, respectively. Then the area S of $\triangle ABC$ is given by

$$\begin{aligned} S &= \frac{1}{2} bc \sin\alpha \\ &= \frac{1}{2} ac \sin\beta \\ &= \frac{1}{2} ab \sin\gamma \\ &= 2R^2 \sin\alpha \sin\beta \sin\gamma \\ &= \frac{1}{2} r(a+b+c) \\ &= \frac{abc}{4R} \\ &= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}. \end{aligned}$$

A property of the common divisor

If c is a common divisor of the integers a and b, then for every pair m, n of integers, c divides $ma + nb$.

Divisibility of polynomials

Let $f(x)$ be a polynomial of degree n.

If $f(a) = 0$, then $f(x) = (x - a)g(x)$ where $g(x)$ is a polynomial whose degree is $n - 1$.

Viète formulae

Let

$$f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$$

be a polynomial of degree n and let $r_1, \ldots, r_n$ be its roots, i.e.

$$f(x) = \prod_{i=1}^{n} (x - r_i).$$

Then

$$a_{n-1} = -\sum_{i=1}^{n} r_i, \quad a_{n-2} = \sum_{\substack{i,j \\ i \neq j}} r_i r_j, \quad \ldots \quad a_0 = (-1)^n \prod_{i=1}^{n} r_i.$$

The Pigeonhole principle

Suppose that the n sets $S_1, \ldots, S_n$ contain m elements, and suppose that $m > kn$ for some positive integer k. Then, at least one of the sets S_i contains more than k elements.

The Arithmetic Mean – Geometric Mean (AM – GM) inequality

Let $a_1, a_2, \ldots, a_n$ be some n real positive numbers. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

and equality is attained if and only if $a_1 = a_2 = \ldots = a_n$.

The fundamental theorem of the algebra

A polynomial of degree n has exactly n complex roots (not necessarily distinct).

Local minimum and local maximum of a function

Let $f(x)$ be a differentiable function in the interval $I = (a, b)$. Then, at each point $c \in I$ where $f(x)$ assumes a local minimum or a local maximum, we have $f'(c) = 0$.

Abel's summation formula (summation by parts)

Let a_i and b_i, $i = 1, 2, \ldots, n$ be two sequences of numbers. Then

$$\sum_{i=1}^{n} a_i b_i = \sum_{j=1}^{n} \left(\sum_{i=1}^{j} a_i \right) (b_j - b_{j+1})$$

where $b_{n+1} \equiv 0$.

Jensen's inequality

Definitions: Let f be a real function defined on the interval I. The function f is called convex on the interval I if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for any $x, y \in I$ and any $0 \leq \lambda \leq 1$.

If f is twice differentiable, and $f''(x) \geq 0$ for all $x \in I$ then f is convex on I (note: this is a sufficient but not necessary condition).

Jensen's inequality: Let f be convex on the interval I. Suppose that $x_1, x_2, \ldots, x_n$ are n points in I and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n nonnegative numbers satisfying $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$. Then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i f(x_i).$$

Euler's formula

Let G be a planar graph (or a polyhedron). Let the number of its faces be f, the number of its edges be e, and the number of its vertices be v. Then,

$$f + v = e + 2$$

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